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Coalition Formation and Data Envelopment Analysis

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Abstract

The introduction of a framework for optimal coalition formation using data envelopment analysis (DEA) methods is the focus of this paper. Simple examples illustrate how DEA is useful in formulating coalition models and deriving optimal solutions. In particular, the paper shows the relevance of the proposed framework in the context of analyzing how companies may reach decisions to acquire potential partners.

Keywords: DEA (data envelopment analysis), coalition formation, core theory, mergers and acquisitions

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We draw on the theory of the core, a topic often taught in connection with general equilibrium theory, to study coalition formation using data envelopment analysis (DEA) methods. In the theory of the core, there is a set {I} of individuals endowed with preferences and initial allocations of resources. A group of {I}, say a subset $S \subset I$, can improve upon its members if each member may increase his or her utility by being a member. The core then consists of the allocation of resources upon which no coalition can improve it.[1]

Decision-making units (DMUs) may enter a coalition with other DMUs to improve their performance (e.g., revenue efficiency or any other appropriate performance measure).[2] Deviating from the standard theory of the core, one can formulate the agents (DMUs) by means of DEA or activity analysis models. Thus, data provided, one can estimate the possible gains of forming coalitions. Having only a finite number of agents, one can estimate the best condition(s) for coalition formation. Note that this paper does not address the allocation of gains among participants in a coalition.

First Example

For the sake of convenience, a simple illustration will aid in introducing the topic of coalition formation and DEA. Consider three DMUs (k = 1,2,3) each using one input (x_k) to produce a single output (y_k). The inputs and outputs are homogeneous, so their sum is well defined, and $\sum_{k=1}^{3} x_k$, $\sum_{k=1}^{3} y_k$ are total input and output, respectively. Table 1 shows how the three DMUs make up the DEA technology.

Table 1
DMU Inputs and Outputs

	DMU 1	DMU 2	DMU 3
Output	y_1	y_2	y_3
Input	x_1	x_2	x_3

Introducing intensity variables, one for each DMU, $z_k \geq 0$ (k = 1,2,3), allows for formulation of a DEA or activity analysis model. In terms of output sets, one could write such a model as follows:

$$P(x) = \left\{ y : \sum_{k=1}^{3} z_k y_k \geqq y, \sum_{k=1}^{3} z_k x_k \leqq x,\ z_k \geqq 0, k = 1,2,3 \right\}.$$

One may prove that this model has strongly disposable input and output (the first two inequalities) and exhibits constant returns to scale, that is

$$P(\lambda x) = \lambda P(x), \quad \lambda > 0.$$

The maximal output that DMU k (k = 1, 2, 3) can produce is estimated as

$$F(x_{k'}) = \text{Max} \left\{ \sum_{k=1}^{3} z_k y_k \ : \sum_{k=1}^{3} z_k x_k \leqq x_{k'},\ z_k \geqq 0,\ k = 1,2,3 \right\}.$$

Its efficiency is the ratio $y_{k'} / F(x_{k'}) \leqq 1,\ k = 1,2,3.$

Allowing DMU 1 and DMU 2 to form a coalition would raise the question of how much output they could jointly produce using their combined amount of input, $x_1 + x_2 = x^{12}$. The output would be

$$F(x_{12}) = \text{Max} \left\{ \sum_{k=1}^{3} z_k^1 y_k + \sum_{k=1}^{3} z_k^2 y_k \ : \sum_{k=1}^{3} z_k^1 x_k \leqq x_1, \sum_{k=1}^{3} z_k^2 x_k \leqq x_2, x_1 + x_2 \leqq x_{12},\ z_k^1 \geqq 0, z_k^2 \geqq 0,\ k = 1,2,3 \right\}.$$

Formulating a coalition between DMU 1 and DMU 2 would be beneficial if

$$F(x_{12}) > F(x_1) + F(x_2).$$

Alternatively, one could formulate a weaker condition for coalition formation in this case as $F(x_{12}) > y_1 + y_2$, where $F(x_1)$ and $F(x_2)$ are the maximal output each DMU can produce not being a member of any coalition, and y_1 and y_2 are the observed outputs.

Similarly, one could form coalitions between DMUs 1 and 3, between DMUs 2 and 3, or among DMUs 1, 2, and 3. Determining the 'best' coalition would involve comparing all the alternatives as follows:

$$F(x_{12}) \text{ vs. } F(x_{13}) \text{ vs. } F(x_{23}) \text{ vs. } F(x_{123}).$$

One could also make weaker comparisons in relation to observed outputs as shown above. The next section involves generalizing these ideas into multi-output multi-input technologies with k finite.

The General Case – Revenue Maximization

Let inputs $x \in R_+^N$, outputs $y \in R_+^M$, and assume there are k = 1, ... , K DMUs (or firms). The constant returns to scale technology may be modeled via DEA or activity analysis as

$$P(x) = \left\{ y : \sum_{k=1}^{K} z_k y_{km} \geqq y_m,\ m = 1,...,M, \sum_{k=1}^{K} z_k x_{kn} \leqq x_n,\ n = 1,...,N,\ z_k \geqq 0,\ k = 1,...,K \right\}$$

where , k = 1, ... , K are the intensity variables forming the convex cone of the observations (x^k, y^k), k = 1, ... , K. Think of x^k as the initial endowment belonging to DMU k, and assume that some of the inputs may be reallocated among the DMUs, say inputs n = 1, ... , N^*. The rest are non-allocable and stay with their DMU. Note that one may have $N^* = N$. Although here each DMU shares the same technology, generated by the data (x^k, y^k), k = 1, ... , K, the DMUs' output set may differ because they may have different initial endowments, x^k (e.g., $x^k \neq x^{n'}$, $k \neq n'$).

In the multi-output formulation, one cannot maximize outputs, so selection of a method that allows for maximization is required. Assuming that output prices (p) are known, one may maximize revenue by maximizing $\sum_{m=1}^{M} p_m y_m$ subject to a technological constraint.

The maximum revenue for DMU k' is

$$R(x^{k'}, p) = \text{Max} \left\{ \sum_{m=1}^{M} p_m y_m \ : \ \sum_{k=1}^{K} z_k y_{km} \geqq y_m,\ m = 1,...,M, \sum_{k=1}^{K} z_k x_{kn} \leqq x_n,\ n = 1,...,N,\ z_k \geqq 0,\ k = 1,...,K \right\}.$$

One may estimate the revenue efficiency for DMU k as the ratio of observed revenue $\sum_{m=1}^{M} p_m y_{k'm} = R(y^{k'})$ to maximum revenue $R(x^{k'}, p)$, that is, $R(y^{k'}) / R(x^{k'}, p)$.

Next, estimate the revenue efficiency gain DMU 1 can make by forming a coalition with, say, DMU 2. Their joint revenue, $R(1,2,p)$, is estimated as follows:

$$R(1,2,p) = \text{Max} \sum_{m=1}^{M} p_m y_m^1 + \sum_{m=1}^{M} p_m y_m^2$$

$s.t.$

$$\sum_{k=1}^{K} z_k^1 y_{km} \geqq y_m^1,\ m = 1,...,M,$$

$$\sum_{k=1}^{K} z_k^1 x_{kn} \leqq \text{x}_n^1,\ n = 1,...,N^*,$$

$$\sum_{k=1}^{K} z_k^1 x_{kn} \leqq \text{x}_{1n},\ n = N^* + 1, \ldots, N,$$

$$z_k^1 \geqq 0, k = 1, \ldots, K,$$

$$\sum_{k=1}^{K} z_k^2 y_{km} \geqq y_m^2, m = 1,\ldots, M,$$

$$\sum_{k=1}^{K} z_k^2 x_{kn} \leqq \mathrm{x}_n^2,\ n = 1,\ldots, N^*,$$

$$\sum_{k=1}^{K} z_k^2 x_{kn} \leqq \mathrm{x}_{2n}, \mathrm{n} = N^* + 1, \ldots, N,$$

$$z_k^2 \geqq 0, k = 1, \ldots, K,$$

$$x_n^1 + x_n^2 \leqq x^{12} = x_{1n} + x_{2n},\ n = 1,\ldots, N^*.$$

Perhaps some further explanation is necessary at this point. Note the following:

1. Each DMU has its own intensity variables, $z_k^1, z_k^2, k = 1, \ldots, K$.
2. Both DMUs face the same output prices; this case can be generalized to $p_k^{k'}$, k = 1, 2.
3. One may reallocate the first n = 1,..., N* inputs to maximize the joint revenue.
4. One can compare the coalition's revenue, $R(1,2,p)$, to individual firm revenue, $R(1,2,p) \geqq R(y^1, p) + R(y^2, p)$, to determine whether a coalition is beneficial.

Evaluating the best coalition option for DMU 1 requires a comparison with all other DMUs, such as DMU 3 through K; for example, (1, 2, 3), (1, 2, 4), and so on. A best coalition exists with K being finite although it need not be unique.

The General Case – Distance Functions Maximization

When data on output prices are not available, one could add directional distance functions. The functions are independent of measurement units and, hence, can be aggregated. They also generalize the first example of adding (scalar) outputs.

The approach is a generalization of Johansen's (1972) industry production model (see Färe & Grosskopf, 2004) and can be viewed as an application of benefit theory due to Luenberger (1995). First, let $P(x)$ be an output set, $P(x) = \{y : x\ can\ produce\ y\}$, and $g \in R_+^M$, $g \neq 0$, a directional vector. The directional output distance function is defined as follows:

$$\vec{D}_o(x, y; g) = \sup\{\beta : (y + \beta g) \in P(x)\}.$$

The directional distance function measures the distance, in the direction of from y to the boundary of the output set; and is a generalization of Shephard's (1970) output distance function,

$$D_o(x, y) = \inf\{\theta : (y/\theta) \in P(x)\},$$

where the relation between the distance functions, for $g = y$, is given by

$$\vec{D}_T(x, y; y) = \frac{1}{D_o(x, y)} - 1.$$

One may estimate the directional output distance function using the DEA or activity analysis formulation of the output set $P(x)$ as

$$\vec{D}_o(x^{k'}, y^{k'}, g) = \sup \beta$$

s.t.

$$\sum_{k=1}^{K} z_k y_{km} \geqq y_{k'm} + \beta g_m,\ m = 1, ..., M,$$

$$\sum_{k=1}^{K} z_k x_{kn} \leqq x_{k'n},\ n = 1, ..., N,$$

$$z_k \geq 0, k = 1, \dots, K.$$

Next, use a distance function criterion to evaluate the benefits of forming a coalition. Paralleling the revenue maximization case, one may calculate the joint directional distance function in the event that DMU 1 and DMU 2 form a coalition as follows:

$$\vec{D}_o(1,2; g) = \text{Max}\, \beta_1 + \beta_2$$

s.t.

$$\sum_{k=1}^{K} z_k^1 y_{km} \geqq y_m^1 + \beta_1 g_m,\ m = 1, ..., M,$$

$$\sum_{k=1}^{K} z_k^1 x_{kn} \leqq x_n^1, n = 1, ..., N^*,$$

$$\sum_{k=1}^{K} z_k^1 x_{kn} \leqq x_{1n}, n = N^* + 1, \dots, N,$$

$$z_k^1 \geqq 0, k = 1, \dots, K,$$

$$\sum_{k=1}^{K} z_k^2 y_{km} \geqq y_m^2 + \beta_2 g_m,\ m = 1, ..., M,$$

$$\sum_{k=1}^{K} z_k^2 x_{kn} \leqq x_n^2, n = 1, ..., N^*,$$

$$\sum_{k=1}^{K} z_k^2 x_{kn} \leqq x_{2n}, n = N^* + 1, \dots, N,$$

$$z_k^2 \geqq 0, k = 1, \dots, K,$$

$$x_n^1 + x_n^2 \leqq x^{12} = x_{1n} + x_{2n},\ n = 1, ..., N^*.$$

Employing the joint distance function, $\vec{D}_o(1,2; g)$, for the two individual DMU distance functions, $\vec{D}_o(x^1, y^1; g)$ and $\vec{D}_o(x^2, y^2; g)$, shows whether a coalition between DMU1 and DMU2 would be beneficial. Again, by evaluating all possible coalitions, one can find the best grouping among DMUs.

Second Example

The second example relates to a case involving strategic choices of companies. In particular, the proposed framework is useful in analyzing how companies may reach decisions to acquire potential partners. Because companies experience increasing difficulty in achieving and sustaining growth, often they resort to forming strategic alliances (e.g., airlines) or acquiring other companies (e.g., the massive waves of mergers and acquisitions activity in the late 1990s). A hypothetical case involving three banks will aid in investigating the issue further. Assume that the banks use two inputs, (personnel) and (capital), to produce a single output, (loans and other investments), as evident in Table 2.

Table 2
Bank Input and Output Data

Bank	A	B	C
y	1	1	1
x_1	2	1	2
x_2	1	2	2

The next question is with which of the other two banks, B or C, Bank A should form a partnership. In this case, allow both inputs to be reallocated. Before committing to a strategy, Bank A must assess the amount of redundant resources that will be a burden should it decide to team up with either Bank B or Bank C. The bank could use surplus resources to achieve economies of scale or alternatively cut costs by eliminating those resources (Dyer, Kale, & Singh, 2004).

To answer the question, one needs to solve two linear programming problems:

LP Problem 1:

$$\text{Max } y_1 + y_2$$

s.t.

$$z_1^1 + z_2^1 + z_3^1 \geqq y_1$$ Bank A

$$z_1^1 2 + z_2^1 + z_3^1 2 \leqq x_{11}$$

$$z_1^1 + z_2^1 2 + z_3^1 2 \leqq x_{12}$$

$$z_k^1 \geqq 0, k = 1, 2, 3$$

$$z_1^2 + z_2^2 + z_3^2 \geqq y_2$$ Bank B

$$z_1^2 2 + z_2^2 + z_3^2 2 \leqq x_{21}$$

$$z_1^2 + z_2^2 2 + z_3^2 2 \leqq x_{22}$$

$$z_k^2 \geqq 0, k = 1, 2, 3$$

$$x_{11} + x_{21} \leqq 3, \; x_{12} + x_{22} \leqq 3$$

LP Problem 2:

$$\text{Max } y_1 + y_3$$

s.t.

$$z_1^1 + z_2^1 + z_3^1 \geqq y_1 \qquad \text{Bank A}$$

$$z_1^1 2 + z_2^1 + z_3^1 2 \leqq x_{11}$$

$$z_1^1 + z_2^1 2 + z_3^1 2 \leqq x_{12}$$

$$z_k^1 \geqq 0, k = 1, 2, 3$$

$$z_1^3 + z_2^3 + z_3^3 \geqq y_3 \qquad \text{Bank C}$$

$$z_1^3 2 + z_2^3 + z_3^3 2 \leqq x_{31}$$

$$z_1^3 + z_2^3 2 + z_3^3 2 \leqq x_{32}$$

$$z_k^3 \geqq 0, k = 1, 2, 3$$

$$x_{11} + x_{31} \leqq 4, \; x_{12} + x_{32} \leqq 3$$

The results show that Bank A should form a partnership with Bank C, not with Bank B. In this case, the total output from Banks A and C is 2.5 units, which is greater than the observed output sum of 2 units produced by any two other banks individually or than the maximum joint output resulting from a potential coalition between Banks A and B, which is also equal to 2 units. The requirement in this example is weaker than in earlier sections, but it illustrates the point.

Summary

The focus of this paper was to propose a framework and present examples demonstrating how one can formulate and estimate optimal coalitions using DEA methods. At the center of such analyses may be cases involving strategic choices of companies (e.g., forming alliances or pursuing takeovers in the interest of boosting sales revenue and profits and maximizing shareholder wealth). Given that corporate history is fraught with a myriad of failed acquisitions and alliances while takeover activity has remained strong as companies experience even more difficulty achieving and sustaining growth, there is strong interest in developing analytical tools to assist companies in making better deals. The proposed framework offers some insights into and tools for helping companies decide whether they should acquire potential partners.

Footnotes

1 In essence, the core is a generalization of the idea of the Pareto set. If an allocation is in the core, every group of agents gets some gain from trade, and no group has an incentive to defect (Varian, 1992).

2 Examples of such coalitions may include corporate alliances and corporate takeovers (Dyer, Kale, & Singh, 2004; Martynova & Renneboog, 2008; Sudarsanam, 2003).

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