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An Introduction to Distributions and Currents

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1. Introduction

These notes are intended as a somewhat vague and preliminary introduction to some mathematical tools which have proven to be very useful in analysis, geometry and dynamics. The concepts explored here are due to Laurent Schwarz and Georges de Rham. The first is responsible for the concept of *distribution* and the latter made enormous contributions to topology and to this end he developed the concept of *current*, and both are in fact very much related. These are subjects which permeate several areas of Mathematics and we chose to give an idea of how this can be of use in the geometric theory of *foliations*. What we present here is not at all a self contained text, on the contrary, we jump from very elementary results to deep ones in the hope that the reader will grasp these ideas and their usefulness.

2. Distributions

2.1. Test functions

2.1.1. The space $\mathcal{D}(U)$

Throughout this section $U \subset \mathbb{R}^n$ will denote a nonempty open set and, unless otherwise stated, $|x|$ is the euclidean norm of $x \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

$C_c^\infty(U)$ denotes the space of infinitely differentiable complex valued functions with compact support defined in U, that is, functions $\phi : U \longrightarrow \mathbb{C}$ with partial derivatives of all orders and such that $supp(\phi) = \overline{\{x \in U \,:\, \phi(x) \neq 0\}}$ is a compact set.

Notation: $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$\partial^\alpha \phi = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \phi.$$

Definition 1. *The space $\mathcal{D}(U)$ of test functions is $C_c^\infty(U)$ together with the following notion of convergence:*

- *a sequence $\phi_j \in C_c^\infty(U)$ converges in $\mathcal{D}(U)$ to the function $\phi \in C_c^\infty(U)$ if, and only if, there is some fixed compact set $K \subset U$ such that supp$(\phi_j - \phi) \subset K$ for all indices j and,*
- *for all multi-indices α, $\partial^\alpha \phi_j \longrightarrow \partial^\alpha \phi$ uniformly on K, that is,*

$$\sup_{x \in K} |\partial^\alpha \phi_j(x) - \partial^\alpha \phi(x)| \to 0 \;\; as \;\; j \to \infty.$$

Remark that $\mathcal{D}(U)$ is a $\mathbb{C}$-vector space.

2.1.2. Bump functions

Prototypes of test functions are bump functions, which we present now.

Exercise 1. Draw the graphs of the following functions:

$f : \mathbb{R} \to \mathbb{R}$, $f(t) = e^{-1/t}$ if $t > 0$, $f(t) = 0$ if $t \leq 0$. Show that $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

$g : \mathbb{R} \to \mathbb{R}$, $g(t) = f(t+2)f(-t-1)$.

$h : \mathbb{R} \to \mathbb{R}$, $h(t) = \dfrac{1}{A} \displaystyle\int_{-\infty}^{t} g(s)\, ds$ where $A = \displaystyle\int_{-\infty}^{\infty} g(s)\, ds$.

Finally set $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi(x) = h(-|x|)$.

Show that all these functions are C^∞. Show that ϕ satisfies $\phi(x) = 0$ if $|x| \geq 2$, $\phi(x) = 1$ if $|x| \leq 1$ and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^n$.

Exercise 2. (*Bump functions*). Given real numbers $0 < a < b$, $\varepsilon > 0$ and $p \in \mathbb{R}^n$ construct a function $\phi_{a,b,\varepsilon,p} : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ such that, $\phi(x) = 0$ for $|x - p| \geq b$, $\phi(x) = \varepsilon$ for $|x - p| \leq a$ and $0 \leq \phi(x) \leq \varepsilon$ for all $x \in \mathbb{R}^n$.

Using the functions $\phi_{a,b,\varepsilon,p}$ we can construct partitions of unity. We shall neither define these objects nor prove their existence, but the interested reader should see G. de Rham's book [5]. However we give a proof of a fact which will be useful in the sequel.

Proposition 2. *Let K be a compact subset of $\mathbb{R}^n$ and U be an open set containing K. There exists a function $\xi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \xi \leq 1$, supp$(\xi) \subset U$ and $\xi = 1$ on an open neighborhood of K.*

Proof. For each point $p \in K$ choose $\rho(p) > 0$ such that the ball $B(p, 3\rho(p)) \subset U$. Since K is compact there are finitely many points p, say $p_1, \ldots, p_m$, such that

$K \subset \bigcup_{j=1}^{m} B(p_j, \rho(p_j))$. Take the functions $\varphi_j = \phi_{\rho(j),2\rho(j),1,p_j}$, $j = 1, \ldots, m$, produced in exercise 2 and define the functions

$$\nu_1 = \varphi_1\,; \quad \nu_{j+1} = \varphi_{j+1} \prod_1^j (1 - \varphi_k)\,; \quad 1 \le j < m. \tag{1}$$

Now, for $j = 1$ the equality

$$\sum_1^j \nu_k = 1 - \prod_1^j (1 - \varphi_k) \tag{2}$$

is obvious. If (2) is true for $j < m$ then, adding ν_{j+1} to (2) and using (1) gives

$$\sum_1^{j+1} \nu_k = \sum_1^j \nu_k + \nu_{j+1} = 1 - \prod_1^j (1 - \varphi_k) + \varphi_{j+1} \prod_1^j (1 - \varphi_k) = 1 - \prod_1^{j+1} (1 - \varphi_k)$$

so that (2) holds for $1 \le j \le m$. Put $\xi = \sum_1^m \nu_k$. ξ is the required function.

□

The function ξ is called a *cut-off function.*

Corollary 3. *Given a point $p \in \mathbb{R}^n$ and a neighborhood U of p, there exists a function $\xi \in C_c^\infty(\mathbb{R}^n)$ such that*

(i) $\xi \ge 0$ *and* $\xi(p) > 0$.

(ii) $supp(\xi) \subset U$.

(iii) $\displaystyle\int_{\mathbb{R}^n} \xi(x)\, dx = 1$.

2.2. A Glimpse on Integration

2.2.1. Measures

Lebesgue integration gives a much more comprehensive theory of distributions than Riemann integration. However, in case the reader is not familiarized with Lebesgue's theory, we advise him (her) to go ahead thinking we are doing Riemann integration (situations at which this will not be possible will be hinted).

The *characteristic function* χ_A of a set $A \subset \mathbb{R}^n$ is defined by:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases} \tag{3}$$

Definition 4. *Let S be a set. A collection Σ of subsets of S is called a sigma-algebra if the following axioms hold:*

- $A \in \Sigma \Longrightarrow A^c = U \setminus A \in \Sigma$.
- *If $A_1, A_2, \dots$ is a countable family of sets in Σ, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.*
- $S \in \Sigma$.

Any family $\mathcal{S}$ of subsets of S can be extended to a sigma-algebra by taking the sigma-algebra of all subsets of S. Now, consider all the sigma-algebras that contain the family $\mathcal{S}$, take their intersection and call it Σ (*Exercise:* Σ is a sigma-algebra.). This is called the *sigma-algebra generated by $\mathcal{S}$.*

An important example of a sigma-algebra is $\mathcal{B}$, the *Borel sigma-algebra*, which is generated by the open subsets of $\mathbb{R}^n$ or, alternatively, generated by the open balls of $\mathbb{R}^n$.

Exercise 3.* Find a subset of $\mathbb{R}^n$ which is NOT contained in $\mathcal{B}$!

Definition 5. *A measure (positive) μ, defined on a sigma algebra Σ, is a function*

$$\mu : \Sigma \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

such that $\mu(\emptyset) = 0$ and which is countably additive, that is, if $A_1, A_2, \dots$ is a sequence of disjoint sets in Σ, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=0}^{\infty} \mu(A_i). \tag{4}$$

A set $A \in \Sigma$ is called *measurable.*

Exercise 4. Show the following properties of measures:

(i) If $A_1, A_2, \dots, A_k$ is a finite collection of disjoint measurable sets then $\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$. This is called finite additivity.

(ii) If $A \subset B$, A and B measurable, then $\mu(A) \leq \mu(B)$.

(iii) If A_i is measurable and $A_1 \subset A_2 \subset A_3 \subset \cdots$, then $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$.

(iv) If A_i is measurable and $A_1 \supset A_2 \supset A_3 \supset \cdots$ and if $\mu(A_1) < \infty$, then $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$.

A *measure space* is then a triple (S, Σ, μ) consisting of a set S, a sigma-algebra Σ and a measure μ. If the set S contains open sets then $\mathcal{B}$ is defined and, if we take $\Sigma = \mathcal{B}$, then μ is called a Borel measure.

Sets of measure zero pose some problems since subsets of sets of measure zero might not be measurable. But there is a procedure, called the *completion* of the measure μ, to eliminate this. We won't need to worry about this. Instead, we will refer to the following property:

Definition 6. *Given a measure space* (S, Σ, μ)*, some property is said to hold* μ-almost everywhere*, or simply* a.e.*, provided the subset of* S *for which the property does not hold is a subset of a set of measure zero.*

Undoubtedly one the most important measures is the Lebesgue measure in $\mathbb{R}^n$. We will not construct it here but we urge the reader to look at its construction in a good book, for instance W. Rudin's [9]. Another useful measure is.

Example 7. *The Dirac delta-measure,* δ_a*, where* $a \in \mathbb{R}^n$ *is a fixed point is defined by*

$$\delta_a(A) = \begin{cases} 1, & \text{if } a \in A \\ 0, & \text{if } a \notin A. \end{cases} \tag{5}$$

Show that δ_a *is a measure. Since it's concentrated in a point, in this case* Σ *could be* $\mathcal{B}$ *or all subsets of* $\mathbb{R}^n$*. Recalling the characteristic function of* A *we have*

$$\delta_a(A) = \chi_A(a). \tag{6}$$

2.2.2. Measurable functions and Integrals

Let $f : U \longrightarrow \mathbb{R}$ be a function on U. Given a sigma-algebra Σ, we say that f is *measurable (with respect to* Σ*)* if, for every $t \in \mathbb{R}$ the set

$$L_f^{>}(t) = \{x \in U : f(x) > t\} \quad \text{is measurable, that is,} \quad L_f^{>}(t) \in \Sigma. \tag{7}$$

A complex function, $f : U \longrightarrow \mathbb{C}$, $f = u + \mathrm{i}v$ is *measurable* provided its real and imaginary parts, u and v, are measurable. Also, we say that a nonnegative measurable function f is a *strictly positive measurable function* on a measurable set A provided $\{x \in A : f(x) = 0\}$ has measure zero, that is, f is positive a.e. (recall Def. 6).

Remark that measurable functions are defined in terms of Σ alone, the existence of a measure is not necessary.

Exercise 5. Show that the following affirmatives are equivalent:

(i) $L_f^{>}(t) \in \Sigma$.

(ii) $L_f^{<}(t) = \{x \in U \,:\, f(x) < t\} \in \Sigma$.

(iii) $L_f^{\geq}(t) = \{x \in U \,:\, f(x) \geq t\} \in \Sigma$.

(iv) $L_f^{\leq}(t) = \{x \in U \,:\, f(x) \leq t\} \in \Sigma$.

Hence we could have used any of these to define measurability of a function. *Hint:*

$$\{x \in U \,:\, f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in U \,:\, f(x) \geq t + 1/n\}.$$

Exercise 6. If $\Sigma = \mathcal{B}$ in $\mathbb{R}^n$ then any continuous or lower semicontinuous or upper semicontinuous function is measurable. *Hint:* f is lower semicontinuous if $L_f^{>}(t)$ is open and upper semicontinuous if $L_f^{<}(t)$ is open.

Exercise 7. Show that, if f and g are measurable, then so are the functions $x \mapsto af(x) + bg(x)$, $a, b \in \mathbb{C}$, $x \mapsto f(x)g(x)$, $x \mapsto |f(x)|$, $x \mapsto h(f(x))$, where $h : \mathbb{C} \longrightarrow \mathbb{C}$ is Borel measurable; $x \mapsto \text{máx}\{f(x), g(x)\}$.

We now define the integral of a measurable function with respect to a measure μ. What we give here is just a very brief sketch; the interested reader should consult a text on the subject, like [7].

Let $f : U \longrightarrow \mathbb{R}_{\geq 0}$ be a nonnegative Σ measurable function. Define

$$R_f(t) = \mu(L_f^{>}(t)).$$

R_f is a nonincreasing function of t since $L_f^{>}(t) \subset L_f^{>}(s)$ whenever $t \geq s$, that is

$$R_f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$$

is a monotone nonincreasing function of t. Therefore its Riemann integral exists (could be ∞) and, by definition,

$$\int_U f(x)\, d\mu(x) = \int_0^{\infty} R_f(t)\, dt. \tag{8}$$

In case f is measurable, nonnegative and $\int_U f\, d\mu < \infty$ we say that f is *summable* or *integrable*.

Suppose now that $f : U \longrightarrow \mathbb{C}$, $f = u + \mathrm{i}v$ with u and v real valued. We split each of u and v as a difference of two nonnegative functions as follows:

$$u(x) = u_+(x) - u_-(x) = \text{máx}\{u(x), 0\} - (-\text{mín}\{u(x), 0\})$$

$$v(x) = v_+(x) - v_-(x) = \text{máx}\{v(x), 0\} - (-\text{mín}\{v(x), 0\}).$$

If f is measurable, then so are u_+, u_-, v_+, v_- and, provided these are summable, we define

$$\int_U f\,d\mu = \int_U u_+\,d\mu - \int_U u_-\,d\mu + \int_U v_+\,d\mu - \int_U v_-\,d\mu. \tag{9}$$

Remark 8. *We shall write* $\int_U f(x)\,dx$ *in case the measure is the Lebesgue measure.*

An important example is that of the characteristic function of a measurable set A [see (3)]. If $\mu(A) < \infty$ then χ_A is summable and

$$\int_A \chi_A\,d\mu = \mu(A). \tag{10}$$

2.3. Distributions

Recall the space $\mathcal{D}(U)$ defined in 2.1.1.

Definition 9. *A distribution is a continuous linear functional on* $\mathcal{D}(U)$*, that is,*

$$T : \mathcal{D}(U) \longrightarrow \mathbb{C}$$

satisfies $T(\lambda\phi) = \lambda T(\phi)$, $T(\phi_1 + \phi_2) = T(\phi_1) + T(\phi_2)$ *for all* $\lambda \in \mathbb{C}$ *and* $\phi, \phi_1, \phi_2 \in \mathcal{D}(U)$.

Continuity means that if $\phi_n \in \mathcal{D}(U)$ *and* $\phi_n \longrightarrow \phi$ *in* $\mathcal{D}(U)$*, then* $T(\phi_n) \longrightarrow T(\phi)$.

The space of distributions on U is denoted by $\mathcal{D}'(U)$. Since distributions can be added and multiplied by complex numbers, this is a $\mathbb{C}$-vector space, *dual* to the space $\mathcal{D}(U)$. Remark that continuity in $\mathcal{D}'(U)$ means sequential continuity.

Convergence of distributions means: a sequence $T_m \in \mathcal{D}'(U)$ *converges in* $\mathcal{D}'(U)$ to $T \in \mathcal{D}'(U)$,

$$T_m \longrightarrow T$$

if, for all $\phi \in \mathcal{D}(U)$, the numerical sequence $T_m(\phi)$ converges to $T(\phi)$,

$$T_m(\phi) \longrightarrow T(\phi).$$

This is indeed a weak kind of convergence.

Now we present examples of distributions that are defined by actual functions.

To do this we introduce the space $L^1_{\text{loc}}(U)$. This consists of the class of functions which are Borel measurable on U and, for each point $a \in U$, there exists an open rectangle $R_a \subset U$, containing a and such that

$$\int_{R_a} |f(x)|\, dx < \infty.$$

We could have used open balls R_a instead of rectangles. These are called *locally summable* or *locally integrable* functions.

To each $f \in L^1_{\text{loc}}(U)$ we define

$$T_f(\phi) = \int\limits_U f(x)\phi(x)\, dx \tag{11}$$

where $\phi \in \mathcal{D}(U)$ is a test function.

Proposition 10. *For every locally summable function f, the map T_f given by (11) defines a distribution on U.*

Proof. T_f is linear since integration is. First we show the absolute convergence of the integral in (11). Since $supp(\phi)$ is compact there are finitely many points $a_1, \dots, a_k$ such that $supp(\phi)$ is contained in the union of the open rectangles $R_1, \dots, R_k$ and $|f|$ is summable in each R_i. We have then $\chi_{supp(\phi)} \leq \chi_{R_1} + \cdots + \chi_{R_k}$. Hence, since $\int\limits_A g(x)\, dx = \int\limits_U \chi_A(x) g(x)\, dx$, $A \subset U$, we get

$$|T_f(\phi)| = \left| \int\limits_{supp(\phi)} f(x)\phi(x)\, dx \right| \leq \int\limits_{supp(\phi)} |f(x)||\phi(x)|\, dx \leq \sum_{i=1}^{k} \int\limits_{R_i} |f(x)||\phi(x)|\, dx \leq$$

$$\leq \left[\sum_{i=1}^{k} \int\limits_{R_i} |f(x)|\, dx \right] \sup_{x \in supp(\phi)} |\phi(x)| = C \sup_{x \in supp(\phi)} |\phi(x)|.$$

Suppose now $\phi_n \longrightarrow \phi$ in $\mathcal{D}(U)$. Hence we have a fixed compact set $K \subset U$ such that $supp(\phi_n - \phi) \subset K$. Then, repeating the estimate above with K in place of $supp(\phi)$ and letting $\tilde{R}_i$ denote appropriate open rectangles covering K we get

$$|T_f(\phi_n) - T_f(\phi)| = |T_f(\phi_n - \phi)| \leq \underbrace{\left[\sum_{i=1}^{k} \int\limits_{\tilde{R}_i} |f(x)|\, dx \right]}_{\text{a constant}} \sup_{x \in K} |\phi_n(x) - \phi(x)| \longrightarrow 0$$

uniformly on K. $\square$

Now we proceed to show that T_f is uniquely determined by the function f. To do this we must introduce *convolutions*. Suppose f and g are complex valued functions in $L^1(\mathbb{R}^n)$, that is, they are measurable and $\int_{\mathbb{R}^n} |f(x)|\,dx < \infty$, $\int_{\mathbb{R}^n} |g(x)|\,dx < \infty$. Their *convolution* is the function $f * g$ defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int_{\mathbb{R}^n} f(z)g(x-z)\,dz. \tag{12}$$

The last equality says $f * g = g * f$ by a change of variables (exercise). Remark that (12) is well defined a.e. and $f * g \in L^1(\mathbb{R}^n)$.

Exercise 8. Let A and B be subsets of $\mathbb{R}^n$ and define

$$A + B = \{a + b \,:\, a \in A, b \in B\}.$$

Show that if A is closed and B is compact then $A + B$ is closed. Give an example with A and B closed and $A + B$ not closed.

Let ϕ be a function like in Corollary 3 and $\epsilon > 0$. Put

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n}\,\phi\left(\frac{x}{\epsilon}\right) \in C_c^\infty(\mathbb{R}^n). \tag{13}$$

Lemma 11. *Suppose $f \in L^1(\mathbb{R}^n)$, f continuous and ϕ_ϵ with supp$(\phi) = \overline{B(0,1)}$. The function $f * \phi_\epsilon$ converges uniformly to f on every compact set K as $\epsilon \to 0$. Moreover, for every $\epsilon > 0$ the function $f * \phi_\epsilon \in C^\infty(\mathbb{R}^n)$.*

Proof. First of all

$$\int_{\mathbb{R}^n} \phi_\epsilon(x)\,dx = \int_{\mathbb{R}^n} \phi\left(\frac{x}{\epsilon}\right)\frac{dx}{\epsilon^n} = \int_{\mathbb{R}^n} \phi(y)\,dy = 1.$$

Then,

$$(f * \phi_\epsilon)(x) - f(x) = \int_{\mathbb{R}^n} f(x-y)\phi_\epsilon(y)\,dy - f(x)\int_{\mathbb{R}^n} \phi_\epsilon(y)\,dy = \int_{\mathbb{R}^n} (f(x-y) - f(x))\phi_\epsilon(y)\,dy =$$

$$= \int_{supp(\phi_\epsilon)} (f(x-y) - f(x))\phi_\epsilon(y)\,dy.$$

This gives

$$|(f * \phi_\epsilon)(x) - f(x)| \leq \int_{supp(\phi_\epsilon)} |f(x-y) - f(x)|\phi_\epsilon(y)\,dy \leq \sup_{y \in supp(\phi_\epsilon)} |f(x-y) - f(x)|.$$

Given $K \subset \mathbb{R}^n$ compact let $K' = K + \overline{B(0,1)}$ which is also compact. Since f is uniformly continuous on any compact set, given $\varepsilon > 0$, there exists a $1 > \delta > 0$ such that $|f(x-y) - f(x)| < \varepsilon$ for all $x \in K \subset K'$ and $|y| < \delta$. Take ϵ small enough so that $supp(\phi_\epsilon) \subset B(0,\delta)$ and conclude the first affirmative. Now,

$$f_\epsilon(x) = f * \phi_\epsilon(x) = \int_{\mathbb{R}^n} f(z)\phi_\epsilon(x-z)\,dz$$

and differentiation under the integral sign gives

$$\partial^\alpha f_\epsilon = \partial^\alpha (f * \phi_\epsilon) = f * \partial^\alpha \phi_\epsilon$$

from which we conclude $f_\epsilon = f * \phi_\epsilon \in C^\infty(\mathbb{R}^n)$. □

Remark 12. *We assumed f is continuous in the lemma above, but this hypothesis is not necessary. However, if we had only $f \in L^1(\mathbb{R}^n)$ then the proof would require more machinery from integration theory than we have at hand. Again, the reader should consult [9]. Anyway, the lemma gives a smooth approximation of a continuous function on any compact set.*

Proposition 13. *Let $C(U)$ be the $\mathbb{C}$-vector space of complex valued continuous functions defined in the open set $U \subset \mathbb{R}^n$. The map*

$$C(U) \longrightarrow \mathcal{D}'(U)$$
$$f \longmapsto T_f$$

is linear and injective.

Proof. Linearity follows immediately from the definition of T_f. Once the map is linear, to show injectivity is enough to show its kernel is the null subspace. Now, f is in the kernel of $f \longmapsto T_f$ if $T_f(\phi) = 0$ for every test function ϕ. Let R be the reflection of a function, $\mathsf{R}(g(z)) = g(-z)$ and T_x be the translation $\mathsf{T}_x(g(z)) = g(z-x)$, so that, for ϵ sufficiently small,

$$(\mathsf{T}_x \mathsf{R} \phi_\epsilon)(z) = \phi_\epsilon(x-z).$$

By (12) we have

$$f_\epsilon(x) = f * \phi_\epsilon(x) = T_f(\mathsf{T}_x \mathsf{R}\phi_\epsilon) \tag{14}$$

But by Lemma 11 f_ϵ converges uniformly to f on compact sets and hence

$$f(x) = \lim_{\epsilon\to 0} f_\epsilon(x) = \lim_{\epsilon\to 0} T_f(\mathsf{T}_x \mathsf{R}\phi_\epsilon) = 0.$$

□

Proposition 13 allows to identify f and T_f whenever $f \in C(U)$.

Remark 14. *In general, for $f \in L^1_{\text{loc}}(U)$ the result is*

$$T_f = T_g \Longleftrightarrow f = g \quad a.e. \tag{15}$$

With this at hand, if T is a distribution in $\mathcal{D}'(U)$ of the form T_f for a function f in $L^1_{\text{loc}}(U)$, then we say that the distribution T is the function f. This allow us to use function notations for distributions, hence, for every $T \in \mathcal{D}'(U)$ we write

$$T(\phi) = \int_U T(x)\phi(x)\,dx,$$

but the reader should have in mind that there is no sense in saying that the distribution T assumes the value $T(x)$ at the point x.

Not every distribution is of the form T_f for some f. A distinguished example is the *Dirac delta-function* (recall (5)):

$$\delta_a(\phi) = \phi(a) \quad \text{for a fixed } a \in U. \tag{16}$$

To see it's not a T_f take a test function ξ as in Corollary 3 with $\xi(a) > 0$. If $\delta_a = T_f$ for some $f \in L^1_{\text{loc}}(U)$ then $f = 0$ a.e. in $U \setminus \{a\}$ and, since $\{a\}$ has meausre zero, $f = 0$ a.e. in U which gives $T_f = 0$, a contradiction since $\delta_a(\xi) \neq 0$.

2.4. Distributions and measures

Let $K \subset U$ be compact. The space $C_c^\infty(K)$ is formed by all $\phi \in C_c^\infty(U)$ such that *supp*$(\phi) \subset K$. For every integer $k \geq 0$ we define the C^k norm in $C_c^\infty(K)$ by

$$\|\phi\|_{(k)} = \sup_{x\in U; |\alpha|\leq k} |\partial^\alpha \phi(x)|. \tag{17}$$

Lemma 15. *Let* $T : C_c^\infty(U) \longrightarrow \mathbb{C}$ *be a linear functional.* $T \in \mathcal{D}'(U)$ *if, and only if, for every compact subset* $K \subset U$*, there exists a constant* $c > 0$ *and* $k \in \mathbb{Z}_{\geq 0}$ *such that*

$$|T(\phi)| \leq c\,\|\phi\|_{(k)} \qquad \forall\ \phi \in C_c^\infty(K). \tag{18}$$

Proof. Recall that (*Definition* 1) $\phi_i \to \phi$ in $C_c^\infty(U)$ if there exists a $K \subset U$, K compact, such that ϕ_i and ϕ belong to $C_c^\infty(K)$ for all i and $\lim\limits_{i\to\infty} \|\phi_i - \phi\| = 0$ for all $k \geq 0$. If (18) holds, then

$$T(\phi_i) - T(\phi) = T(\phi_i - \phi) \longrightarrow 0 \quad \text{as } i \to \infty.$$

Suppose now T does not satisfy the hypothesis above. Then we can find a compact set $K \subset U$ such that, for every $c > 0$ and $k \geq 0$ there exists $\phi_{c,k} \in C_c^\infty(K)$ satisfying $|T(\phi_{c,k})| > c\,\|\phi_{c,k}\|_{(k)}$. Put $\varphi_{c,k} = \dfrac{1}{|T(\phi_{c,k})|}\phi_{c,k}$. This gives

$$\|\varphi_{c,k}\|_{(k)} = \frac{1}{|T(\phi_{c,k})|}\|\phi_{c,k}\|_{(k)} < \frac{1}{c} \qquad \text{and } |T(\varphi_{c,k})| = 1.$$

But then

$$\lim_{k\to\infty} \varphi_{k,k} \to 0 \quad \text{in } C_c^\infty(U) \quad \text{and } T(\varphi_{k,k}) \nrightarrow 0$$

and T is not continuous, hence is not a distribution. □

Definition 16. *Let* $T \in \mathcal{D}'(U)$*. The smallest integer* k *for which (18) holds, for some constant* c*, is the order of* T *on* K*. The supremum over all compact sets* $K \subset U$ *of the orders of* T *on* K *is the order of the distribution* T *on* U*.*

Exercise 9. (i) Show that $T(\phi) = \phi^{(k)}(0)$, $\phi \in C_c^\infty(\mathbb{R})$ is a distribution in $\mathcal{D}'(\mathbb{R})$ of order $\leq k$.

(ii) Show that T as in (i) is of order k in any neighborhood of 0. *Hint:* Consider $\phi_\epsilon(x) = x^k\varphi(x/\epsilon)$ where φ is a bump function equal to 1 around 0.

(iii) Show that the distribution given by $S(\phi) = \sum\limits_0^\infty (-1)^k\phi^{(k)}(k)$ is not of finite order.

We now consider distributions of finite order k and show they can be identified with continuous linear forms on the space $C_c^k(U)$, which is the space of functions of class C^k with compact support in U.

First, *convergence* in $C_c^k(U)$ follows Definition 1 with the difference that only multi-indices α with $|\alpha| \leq k$ are considered. Also remark that, if $\infty \geq k > m$ then, a sequence $\phi_i \in C_c^k(U)$ converges in $C_c^k(U)$ to ϕ implies that ϕ_i also converges to ϕ in $C_c^m(U)$. Since $\mathcal{D}(U) \subsetneq C_c^m(U)$ the restriction to $\mathcal{D}(U)$ of a continuous linear form on $C_c^m(U)$ is a distribution of order $\leq m$.

Proposition 17. *Let $T \in \mathcal{D}'(U)$ be a distribution of order $\leq k$. Then T extends in a unique way to a continuous linear form S on $C_c^k(U)$.*

Proof. Recall the functions ϕ_ϵ of (13). If $f \in C_c^k(U)$ then the functions $f_\epsilon = f * \phi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ satisfy $\partial^\alpha f_\epsilon = \partial^\alpha f * \phi_\epsilon$ for all α with $|\alpha| \leq k$ and $\partial^\alpha f_\epsilon \to \partial^\alpha f$ uniformly as $\epsilon \to 0$. Now, for all sufficiently small $\epsilon > 0$, the supports $supp(f_\epsilon)$ are contained in a fixed compact subset of U (exercise) and then $f_\epsilon \in C_c^\infty(U)$ converge to $f \in C_c^k(U)$. We have

$$|T(f_\epsilon) - T(f_\rho)| = |T(f_\epsilon - f_\rho)| \leq c\|f_\epsilon - f_\rho\|_{(k)}$$

and $\|f_\epsilon - f_\rho\|_{(k)} \to 0$ as $\epsilon, \rho \to 0$, which says that $\epsilon \mapsto T(f_\epsilon)$ is a Cauchy sequence in $\mathbb{C}$ and hence it has a limit $S(f) \in \mathbb{C}$. Since $f \mapsto f * \phi_\epsilon$ is linear we have defined a linear form S on $C_c^k(U)$. Moreover

$$|S(f)| \leq |S(f) - T(f_\epsilon)| + |T(f_\epsilon)| \leq |S(f) - T(f_\epsilon)| + c\|f_\epsilon\|_{(k)}$$

$$\leq |S(f) - T(f_\epsilon)| + c\|f\|_{(k)} + c\|f - f_\epsilon\|_{(k)} \to c\|f\|_{(k)} \text{ as } \epsilon \to 0$$

and, as $|S(f)| \leq c\|f\|_{(k)}$, S is continuous on $C_c^k(U)$ and we also have $S(f) = T(f)$ for $f \in C_c^\infty(U)$. Remark that S is unique (limit of a number sequence) and does not depend on the choice of the family of functions ϕ_ϵ. □

We identify T with S, its extension to $C_c^k(U)$ and write $T = S$.

A *Radon measure* is a continuous linear form on the space $C_c^0(U)$ of continuous functions with compact support in the open set $U \subset \mathbb{R}^n$. By Proposition 10 (and Remark 14) T_f is a distribution of order 0, hence extends to a unique Radon measure on U called the measure with *density function f*.

Definition 18. *A distribution $T \in \mathcal{D}'(U)$ is positive if, for all $\phi \in \mathcal{D}(U)$ with $\phi \geq 0$ we have $T(\phi) \geq 0$. A measure T on U is positive if $T(f) \geq 0$ for all non-negative $f \in C_c^0(U)$. We write $T \geq 0$ if T is positive.*

We close this first section with the

Theorem 19. *A positive distribution is a positive measure.*

Proof. Let $T \in \mathcal{D}'(U)$ be a positive distribution. Given a compact set $K \subset U$, let $\varrho \in \mathcal{D}(U)$ be such that $\varrho = 1$ on K and $0 \leq \varrho \leq 1$ (Proposition 2). This gives, by hypothesis, $T(\varrho) \geq 0$. Let $\phi \in C_c^\infty(K)$ be real-valued and $c = \|\phi\| = \sup_{x\in K} |\phi(x)|$. Then $c\varrho - \phi \geq 0$ and

$$cT(\varrho) - T(\phi) = T(c\varrho - \phi) \geq 0.$$

It follows that $cT(\varrho) - T(\phi)$ is real and non-negative. Hence $T(\phi)$ is real and

$$T(\phi) \leq T(\varrho)\|\phi\|.$$

If $\phi \in C_c^\infty(K)$ is complex valued then $T(\phi) = T(\mathrm{Re}\,\phi) + \mathrm{i}\,T(\mathrm{Im}\,\phi)$. Write $\theta = \arg T(\phi)$ and $\varphi = \mathrm{Re}\,(e^{-\mathrm{i}\,\theta}\phi)$. Then,

$$|T(\phi)| = e^{-\mathrm{i}\,\theta}T(\phi) = T(e^{-\mathrm{i}\,\theta}\phi)$$

hence $T(e^{-\mathrm{i}\,\theta}\phi)$ is real since it equals $|T(\phi)|$, a non-negative real number. But then

$$T(e^{-\mathrm{i}\,\theta}\phi) = T(\mathrm{Re}\,(e^{-\mathrm{i}\,\theta}\phi)) = T(\varphi)$$

and we get

$$|T(\phi)| = T(\varphi) \leq T(\varrho)\|\varphi\| \leq T(\varrho)\|\phi\|.$$

This shows T has order 0 and, by Proposition 17, it extends to a unique continuous linear form T on $C_c^0(U)$.

Now, if $f \in C_c^0(U)$ with $f \geq 0$, then $f_\epsilon = f * \phi_\epsilon \geq 0$ since $\phi_\epsilon \geq 0$. It follows that $T(f_\epsilon) \geq 0$ and, as $\lim\limits_{\epsilon\to 0} T(f_\epsilon) = T(f)$, that $T(f) \geq 0$, hence T is a positive measure. □

Remark 20. *In fact this Theorem can be rephrased as follows:*

Let $T \in \mathcal{D}'(U)$ be a positive distribution. Then, there is a unique positive Borel measure μ on U such that $\mu(K) < \infty$ for all compact sets $K \subset U$ and such that, for all $\phi \in \mathcal{D}(U)$

$$T(\phi) = \int_U \phi(x)d\mu(x). \tag{19}$$

Conversely, any positive Borel measure with $\mu(K) < \infty$ for all compact sets $K \subset U$ defines a positive distribution via (19).

We do not present a proof of this result because it involves the construction of measures from outer measures. It is an extension of the so-called Riez-Markov representation theorem (see [9]).

2.5. Derivatives

Recall that $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$\partial^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f\,, \quad \partial_i f = \frac{\partial f}{\partial x_i}.$$

If f is a C^1 function on the open set $U \subset \mathbb{R}^n$ then integration by parts gives

$$\int_U \partial_i f(x)\, \phi(x)\, dx = -\int_U f(x)\, \partial_i \phi(x)\, dx$$

where ϕ is a test function (exercise), which tells us that

$$T_{\partial_i f}(\phi) = -T_f(\partial_i \phi).$$

We use this to define derivatives of distributions.

Definition 21. *If $T \in \mathcal{D}'(U)$ and α is a multi-index, the distributional or weak derivative $\partial^\alpha T$ is defined by*

$$\partial^\alpha T(\phi) = (-1)^\alpha\, T(\partial^\alpha \phi). \tag{20}$$

Lemma 22. *$\partial^\alpha T$ is a distribution.*

Proof. Since differentiation is linear we must only show its continuity on $\mathbb{D}(U)$. First we show that if $\phi \in \mathcal{D}(U)$ then so does $\partial^\alpha \phi$ for all α. So, suppose $\phi_i \to \phi$ in $\mathcal{D}(U)$. Then *supp*$(\partial^\alpha \phi_i - \partial^\alpha \phi) \subset$ *supp*$(\phi_i - \phi) \subset K \subset U$. Let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index. Since $\partial^\beta(\partial^\alpha \phi_j) - \partial^\beta(\partial^\alpha \phi) = \partial^{\beta+\alpha}\phi_j - \partial^{\beta+\alpha}\phi$ we have that $\partial^{\beta+\alpha}\phi_j - \partial^{\beta+\alpha}\phi$ converges uniformly to zero on compact sets and hence $\partial^\alpha \phi \in \mathcal{D}(U)$. Thus,

$$(-1)^\alpha\, T(\partial^\alpha \phi_j) \to (-1)^\alpha\, T(\partial^\alpha \phi) \quad \text{as} \quad j \to \infty$$

and this is the same as $\partial^\alpha T(\phi_j) \to \partial^\alpha T(\phi)$ and we are done. □

Lemma 23. *$\partial^\alpha : \mathcal{D}'(U) \longrightarrow \mathcal{D}'(U)$ is a continuous map, for all α.*

Proof. Suppose $\lim\limits_{j\to\infty} T_j(\phi) = T(\phi)$ for all $\phi \in \mathcal{D}(U)$. Since the $\partial^\alpha \phi$ are test functions we get, by definition of differentiation,

$$\lim_{j\to\infty} \partial^\alpha T_j(\phi) = \lim_{j\to\infty} (-1)^\alpha T_j(\partial^\alpha \phi) = (-1)^\alpha T(\partial^\alpha \phi) = \partial^\alpha T(\phi).$$

□

Remark that this notion of weak derivative extends the usual notion of derivative and agrees with it provided the usual derivative exists and is continuous. In this weak sense, all distributions have derivatives of all orders. However, the distributional derivative of a non-differentiable function (in the usual sense) is not necessarily a function, as is shown in the following.

Example 24.

The Heaviside function is defined as the distribution associated to the characteristic function of $[0,\infty)\subset\mathbb{R}$, $H=\chi_{[0,\infty)}\in\mathcal{D}'(\mathbb{R})$. Its derivative is

$$H'(\phi) = -H(\phi') = -\int_0^\infty \phi'(x)\,dx = -(0-\phi(0)) = \phi(0) \quad \forall\phi\in\mathcal{D}(\mathbb{R}). \tag{21}$$

Hence, $H'=\delta_0$, the Dirac delta function at 0. In general, given an interval $[a,b]$, $a<b$, its characteristic function, seen as a distribution, can be writen as $\chi_{[a,b]}=\mathsf{T}_\mathsf{a}H-\mathsf{T}_\mathsf{b}H$ where $\mathsf{T}_x H(y)=H(y-x)$. It follows that

$$\chi'_{[a,b]} = \delta_a - \delta_b.$$

3. Manifolds

3.1. Definitions

A *complex manifold* (C^k, C^∞, C^ω = real analytic) of dimension n is a topological space M, which is Hausdorff, connected and with a countable basis, endowed with an analytic structure defined as follows: there exists an open covering $\{U_\alpha\}_{\alpha\in A}$ of M and homeomorphisms $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$ where $V_\alpha\subset\mathbb{C}^n$ ($V_\alpha\subset\mathbb{R}^n$) is open, such that the *changes of coordinates*

$$\varphi_\alpha\circ\varphi_\beta^{-1} \tag{22}$$

are *holomorphic* (C^k, C^∞, C^ω) where defined. φ_α is called a *chart* and, for $z\in M$, $\varphi_\alpha(z)=(z_1^\alpha,\ldots,z_n^\alpha)\in\mathbb{C}^n$ are called the *local coordinates* in U_α. The collection $\{U_\alpha,\varphi_\alpha\}$ is called a holomorphic (C^k, C^∞, C^ω) *atlas* for M.

If M has dimension n, a connected subset $N\subset M$ is a *submanifold* of dimension $m\le n$ if, for each $z\in N$ there exists a chart $\{U_\alpha,\varphi_\alpha\}$, with $z\in U_\alpha$, such that φ_α is a homeomorphism between $U_\alpha\cap N$ and an open set of $\mathbb{C}^m\times\{0\}\subset\mathbb{C}^m\times\mathbb{C}^{n-m}\cong\mathbb{C}^n$.

Given manifolds M and N, a map $f : M \longrightarrow N$ is *holomorphic* (C^k, C^∞, C^ω) provided the compositions

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} \tag{23}$$

are holomorphic (C^k, C^∞, C^ω) where defined, with ψ_β and φ_α charts in N and M respectively.

$X \subset M$ is an *analytic set* if, for each $z \in M$ there is an open neighborhood $U \subset M$ of z and a holomorphic map $f : U \longrightarrow \mathbb{C}^\ell$ such that $X \cap U = f^{-1}(0)$ (ℓ may depend on z).

If $W \subset M$ is open and $\ell \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ then $C^\ell(W, \mathbb{C})$ ($C^\ell(W, \mathbb{R})$) is the space of functions of class C^ℓ in W. In case W is not open, it is the space of functions which admit a C^ℓ extension to a neighborhood of W.

3.2. Tangent spaces

A complex manifold M of dimension n is naturally a real analytic manifold of dimension $2n$. Given a point $p \in M$ take a chart $\{U_\alpha, \varphi_\alpha\}$ in M with $p \in U_\alpha$. The coordinates of the points in U_α are

$$\begin{aligned} \varphi_\alpha(q) &= (z_1^\alpha(q), \ldots, z_n^\alpha(q)) \in \mathbb{C}^n \\ &= (x_1^\alpha(q) + \mathrm{i}y_1^\alpha(q), \ldots, x_n^\alpha(q) + \mathrm{i}y_n^\alpha(q)) \\ &= (x_1^\alpha(q), y_1^\alpha(q), \ldots, x_n^\alpha(q), y_n^\alpha(q)) \in \mathbb{R}^{2n} \sim \mathbb{C}^n. \end{aligned}$$

The *real tangent space* of M at a point p, T_pM, is the space of functions of class C^∞ (called derivations) $\nu : M \longrightarrow \mathbb{R}$ satisfying:

(i) ν is $\mathbb{R}$-linear and

(ii) $\nu(fg) = g(p)\nu(f) + f(p)\nu(g)$ (Leibniz's rule).

ν is called a *tangent vector* at p. If $f \in C^\infty(M, \mathbb{R})$ then, by definition,

$$\frac{\partial f}{\partial x_i^\alpha}(p) = \frac{\partial (f \circ \varphi_\alpha^{-1})}{\partial x_i^\alpha}(\varphi_\alpha(p)) \quad \text{and similarly for the } y_i^\alpha s.$$

Hence, $\dfrac{\partial}{\partial x_i^\alpha}(p)$ is a tangent vector at z and

$$\left\{ \frac{\partial}{\partial x_1^\alpha}(p), \frac{\partial}{\partial y_1^\alpha}(p), \ldots, \frac{\partial}{\partial x_n^\alpha}(p), \frac{\partial}{\partial y_n^\alpha}(p) \right\}$$

is a real basis of T_pM (exercise).

Complexify T_pM, that is, $T_pM^{\mathbb{C}} = T_pM \otimes \mathbb{C}$ which means: simply allow multiplication by complex numbers. This is a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} T_pM^{\mathbb{C}} = 2n$. For $p \in U_\alpha$, choose for $T_pM^{\mathbb{C}}$ the basis

$$\left\{\frac{\partial}{\partial z_1^\alpha}(p), \frac{\partial}{\partial \bar{z}_1^\alpha}(p), \dots, \frac{\partial}{\partial z_n^\alpha}(p), \frac{\partial}{\partial \bar{z}_n^\alpha}(p)\right\}$$

where

$$\frac{\partial}{\partial z_k^\alpha}(p) = \frac{1}{2}\left(\frac{\partial}{\partial x_k^\alpha}(p) - \mathrm{i}\frac{\partial}{\partial y_k^\alpha}(p)\right) \text{ and } \frac{\partial}{\partial \bar{z}_k^\alpha}(p) = \frac{1}{2}\left(\frac{\partial}{\partial x_k^\alpha}(p) + \mathrm{i}\frac{\partial}{\partial y_k^\alpha}(p)\right) \tag{24}$$

Let's examine changes of coordinates (22) in more detail. Set

$$\widetilde{\Theta}_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$$

and write

$$\widetilde{\Theta}_{\alpha\beta}(x_1, y_1, \dots, x_n, y_n) = (u_1, v_1, \dots, u_n, v_n)$$

(real coordinates). The derivative of $\widetilde{\Theta}_{\alpha\beta}$ is given by the matrix

$$D\widetilde{\Theta}_{\alpha\beta} = \begin{pmatrix} \dfrac{\partial(u_1, v_1)}{\partial(x_1, y_1)} & \cdots & \dfrac{\partial(u_n, v_n)}{\partial(x_n, y_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial(u_n, v_n)}{\partial(x_1, y_1)} & \cdots & \dfrac{\partial(u_n, v_n)}{\partial(x_n, y_n)} \end{pmatrix}$$

where

$$\frac{\partial(u_j, v_j)}{\partial(x_k, y_k)} = \begin{pmatrix} \dfrac{\partial u_j}{\partial x_k} & \dfrac{\partial u_j}{\partial y_k} \\ \dfrac{\partial v_j}{\partial x_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix}, \quad 1 \le j, k \le n.$$

Now write $\widetilde{\Theta}_{\alpha\beta} = (\widetilde{\Theta}_1, \dots, \widetilde{\Theta}_n)$ where $\widetilde{\Theta}_j = u_j + \mathrm{i}v_j$. Changing from the basis

$$\left\{\frac{\partial}{\partial x_1}(z), \frac{\partial}{\partial y_1}(z), \dots, \frac{\partial}{\partial x_n}(z), \frac{\partial}{\partial y_n}(z)\right\}$$

to the basis

$$\left\{\frac{\partial}{\partial z_1}(z), \frac{\partial}{\partial \bar{z}_1}(z), \dots, \frac{\partial}{\partial z_n}(z), \frac{\partial}{\partial \bar{z}_n}(z)\right\}$$

Exercise 10. Show that the change from the basis $\{\partial/\partial x_j, \partial/\partial y_j\}$ to the basis $\{\partial/\partial z_j, \partial/\partial \bar{z}_j\}$ is given by the matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ -\mathrm{i}/2 & \mathrm{i}/2 \end{pmatrix} \text{ with } P^{-1} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}.$$

◁

the matrix representing $D\widetilde{\Theta}_{\alpha\beta}$ becomes

$$\begin{pmatrix} P^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P^{-1} \end{pmatrix} \begin{pmatrix} \dfrac{\partial(u_1,v_1)}{\partial(x_1,y_1)} & \cdots & \dfrac{\partial(u_1,v_1)}{\partial(x_n,y_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial(u_n,v_n)}{\partial(x_1,y_1)} & \cdots & \dfrac{\partial(u_n,v_n)}{\partial(x_n,y_n)} \end{pmatrix} \begin{pmatrix} P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{\partial\widetilde{\Theta}_1}{\partial z_1} & \dfrac{0}{\dfrac{\partial\widetilde{\Theta}_1}{\partial z_1}} & \cdots & \dfrac{\partial\widetilde{\Theta}_1}{\partial z_n} & \dfrac{0}{\dfrac{\partial\widetilde{\Theta}_1}{\partial z_n}} \\ 0 & & & 0 & \\ & \vdots & \ddots & \vdots & \\ \dfrac{\partial\widetilde{\Theta}_n}{\partial z_1} & \dfrac{0}{\dfrac{\partial\widetilde{\Theta}_n}{\partial z_1}} & \cdots & \dfrac{\partial\widetilde{\Theta}_n}{\partial z_n} & \dfrac{0}{\dfrac{\partial\widetilde{\Theta}_n}{\partial z_n}} \\ 0 & & & 0 & \end{pmatrix}.$$

Now, changing from the basis

$$\left\{ \frac{\partial}{\partial z_1}(z), \frac{\partial}{\partial \bar{z}_1}(z), \ldots, \frac{\partial}{\partial z_n}(z), \frac{\partial}{\partial \bar{z}_n}(z) \right\}$$

to the basis

$$\left\{ \frac{\partial}{\partial z_1}(z), \ldots \frac{\partial}{\partial z_n}(z), \ldots, \frac{\partial}{\partial \bar{z}_1}(z), \frac{\partial}{\partial \bar{z}_n}(z) \right\}$$

the matrix above becomes

$$\begin{pmatrix} \dfrac{\partial\widetilde{\Theta}_1}{\partial z_1} & \cdots & \dfrac{\partial\widetilde{\Theta}_1}{\partial z_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial\widetilde{\Theta}_n}{\partial z_1} & \cdots & \dfrac{\partial\widetilde{\Theta}_n}{\partial z_n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \overline{\dfrac{\partial\widetilde{\Theta}_1}{\partial z_1}} & \cdots & \overline{\dfrac{\partial\widetilde{\Theta}_1}{\partial z_n}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \overline{\dfrac{\partial\widetilde{\Theta}_n}{\partial z_1}} & \cdots & \overline{\dfrac{\partial\widetilde{\Theta}_n}{\partial z_n}} \end{pmatrix},$$

so that the derivative $D\widetilde{\Theta}_{\alpha\beta}$ has the matrix

$$D\widetilde{\Theta}_{\alpha\beta} = \begin{pmatrix} \Theta_{\alpha\beta} & 0 \\ 0 & \overline{\Theta}_{\alpha\beta} \end{pmatrix}$$

where

$$\Theta_{\alpha\beta} = \left(\frac{\partial\widetilde{\Theta}_i}{\partial z_j}\right)_{1\leq i,j\leq n}$$

Hence,

$$\det D\widetilde{\Theta}_{\alpha\beta} = \det\Theta_{\alpha\beta}\det\overline{\Theta}_{\alpha\beta} = |\det\Theta_{\alpha\beta}|^2 > 0. \tag{25}$$

This means that complex manifolds are (naturally) *orientable*. For the definition of *orientability* see [5].

We use this last basis to decompose $T_pM^{\mathbb{C}}$ into 2 subspaces:

$$T_p'M = \left\langle \frac{\partial}{\partial z_1}(p), \dots \frac{\partial}{\partial z_n}(p)\right\rangle_{\mathbb{C}} \tag{26}$$

the holomorphic tangent space and

$$T_p''M = \left\langle \frac{\partial}{\partial \bar{z}_1}(p), \dots, \frac{\partial}{\partial \bar{z}_n}(p)\right\rangle_{\mathbb{C}} \tag{27}$$

the anti-holomorphic tangent space, so

$$T_pM^{\mathbb{C}} = T_p'M \oplus T_p''M. \tag{28}$$

The real tangent bundle of M, TM, is the union $TM = \bigcup_{p\in M} T_pM$. Now, in local coordinates, since a tangent vector to M at a point is identified with a vector in $\mathbb{R}^{2n}$,

$$TU_\alpha \;=\; \bigcup_{p\in U_\alpha} T_pM \;=\; \{(p, v_\alpha) : p \in U_\alpha, v_\alpha \in \mathbb{R}^{2n}\},$$

and we conclude that TU_α has a product structure $U_\alpha \times \mathbb{R}^{2n}$. Hence,

$$TM \;=\; \bigcup_\alpha TU_\alpha \;=\; \bigcup_\alpha U_\alpha \times \mathbb{R}^{2n},$$

where, for $p \in U_\alpha \cap U_\beta$, (p, v_α) e (p, v_β) are the same point of TM if, and only if,

$$v_\alpha = D(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(p))v_\beta. \tag{29}$$

It follows that TM is a real analytic manifold obtained by gluing the $U_\alpha \times \mathbb{R}^{2n}$ by means of the identification given in (29). The changes of coordinates, or transition functions for TM are given by

$$\Phi_{\alpha\beta} \;=\; (\varphi_\alpha \circ \varphi_\beta^{-1}, D(\varphi_\alpha \circ \varphi_\beta^{-1})). \tag{30}$$

Also, since the projections $U_\alpha \times \mathbb{R}^{2n} \xrightarrow{\pi_\alpha} U_\alpha$ and $U_\beta \times \mathbb{R}^{2n} \xrightarrow{\pi_\beta} U_\beta$ coincide in the intersection $U_\alpha \times \mathbb{R}^{2n} \cap U_\beta \times \mathbb{R}^{2n}$, the projection $\pi : TM \to M$ given locally by $(p, v_\alpha) \mapsto p$ is well defined.

Since $TM^{\mathbb{C}} = TM \otimes \mathbb{C}$ we deduce from (28) a decomposition

$$TM^{\mathbb{C}} = T'M \oplus T''M \tag{31}$$

where $T'M$ is the *holomorphic tangent bundle* of M and $T''M$ is the *anti-holomorphic tangent bundle* of M. The transition functions of $T'M$ are

$$(\varphi_\alpha \circ \varphi_\beta^{-1}, \Theta_{\alpha\beta})$$

and those of $T''M$

$$(\varphi_\alpha \circ \varphi_\beta^{-1}, \overline{\Theta}_{\alpha\beta}).$$

If TM^* is the real cotangent bundle then $TM^{\mathbb{C}*} = T'M^* \oplus T''M^*$ are the corresponding cotangent bundles, with transition functions given respectively by

$$\left(\varphi_\alpha \circ \varphi_\beta^{-1}, \left((\Theta_{\alpha\beta} \oplus \overline{\Theta}_{\alpha\beta})^T\right)^{-1}\right), \; \left(\varphi_\alpha \circ \varphi_\beta^{-1}, (\Theta_{\alpha\beta}^T)^{-1}\right), \; \left(\varphi_\alpha \circ \varphi_\beta^{-1}, (\overline{\Theta}_{\alpha\beta}^T)^{-1}\right).$$

Let $f : M^n \to N^n$ be a holomorphic map, see (23). The matrix of the derivative

$$Df(p) : T_pM^{\mathbb{C}} \to T_{f(p)}N^{\mathbb{C}}$$

with respect to the bases

$$\left\{\frac{\partial}{\partial z_1}(p), \dots \frac{\partial}{\partial z_n}(p), \dots, \frac{\partial}{\partial \bar{z}_1}(p), \frac{\partial}{\partial \bar{z}_n}(p)\right\}$$

of $T_pM^{\mathbb{C}}$ and

$$\left\{\frac{\partial}{\partial z'_1}(f(p)), \dots \frac{\partial}{\partial z'_n}(f(p)), \dots, \frac{\partial}{\partial \bar{z'}_1}(f(p)), \frac{\partial}{\partial \bar{z'}_n}(f(p))\right\}$$

of $T_{f(p)}N^{\mathbb{C}}$ is given by

$$Df(p) = \begin{pmatrix} \dfrac{\partial f_i}{\partial z_j}(p) & 0 \\ & \\ 0 & \overline{\dfrac{\partial f_i}{\partial z_j}}(p). \end{pmatrix}$$

In particular, if $Df(p)$ is an isomorphism then,

$$\det Df(p) \;=\; \left|\det\left(\frac{\partial f_i}{\partial z_j}(p)\right)\right|^2 > 0,$$

that is, holomorphic diffeomorphisms preserve orientations.

3.2.1. Examples

The most simple example of a complex manifold is $\mathbb{C}^n$, $n \geq 1$. We will digress on an important example, that of the projective spaces.

The complex projective space of dimension n, $\mathbb{P}^n_{\mathbb{C}}$, is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ under the identification

$$z \sim w \iff \exists\, \lambda \in \mathbb{C}^* \text{ such that } z = \lambda w.$$

The class of a point z is denoted by $[z]$ or $(z_0 : z_1 : \cdots : z_n)$ and the quotient map $\mathbb{C}^{n+1} \to \mathbb{P}^n_{\mathbb{C}}$ is denoted by P. We provide $\mathbb{P}^n_{\mathbb{C}}$ with the topology induced by P, which makes it a compact space (exercise). Besides, $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold with the atlas defined by $\{U_i, \varphi_i\}$, $i = 0, \dots, n$, $U_i = \{[z] \in \mathbb{P}^n_{\mathbb{C}} : z_i \neq 0\}$, where $\varphi_i : U_i \to \mathbb{C}^n$ is given by

$$\varphi_i(z_0 : z_1 : \cdots : z_n) \;=\; \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i}\right),$$

where " $\widehat{\ }$ " means omission. To avoid heavy notation let us consider φ_0 and φ_j. We have

$$\begin{aligned}\varphi_0(z_0 : z_1 : \cdots : z_n) &= \left(\widehat{1}, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) = (x_1, \ldots, x_n),\\ \varphi_j(z_0 : z_1 : \cdots : z_n) &= \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \widehat{1}, \frac{z_{j+1}}{z_j} \ldots, \frac{z_n}{z_j}\right)\\ &= (y_1, \ldots, y_j, \widehat{1}, y_{j+1}, \ldots, y_n).\end{aligned}$$

Let $U_{0j} = U_0 \cap U_j$. The change of coordinates $U_{0j} \xrightarrow{\varphi_j \circ \varphi_0{}^{-1}} U_{0j}$ is given by

$$\varphi_j \circ \varphi_0{}^{-1}(x_1, \ldots, x_n) = \left(\frac{1}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

The derivative $D(\varphi_j \circ \varphi_0{}^{-1}) = \Theta_{j0} : U_{0j} \to GL(n, \mathbb{C})$ is represented by the matrix

$$\Theta_{j0} = \begin{pmatrix} 0 & 0 & \ldots & 0 & -\frac{1}{x_j^2} & 0 & \ldots & 0 & 0\\ \frac{1}{x_j} & 0 & \ldots & 0 & -\frac{x_1}{x_j^2} & 0 & \ldots & 0 & 0\\ 0 & \frac{1}{x_j} & \ldots & 0 & -\frac{x_2}{x_j^2} & 0 & \ldots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \ldots & 0 & -\frac{x_{n-1}}{x_j^2} & 0 & \ldots & \frac{1}{x_j} & 0\\ 0 & 0 & \ldots & 0 & -\frac{x_n}{x_j^2} & 0 & \ldots & 0 & \frac{1}{x_j} \end{pmatrix}. \tag{32}$$

In particular, $\det \Theta_{j0} = (-1)^j \,(1/x_j)^{n+1} = (-1)^j \,(z_0/z_j)^{n+1}$. More generally $D(\varphi_i \circ \varphi_j^{-1}) = \Theta_{ij}$ satisfies $\det \Theta_{ij} = (-1)^{i+j} \,(z_j/z_i)^{n+1}$.

A piece of notation. When we take local coordinates in $U_i = \{z_i \neq 0\}$ in $\mathbb{P}^n_{\mathbb{C}}$, that is, $(z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n)$ we cover all of $\mathbb{P}^n_{\mathbb{C}}$ except the set $\{(z_0 : \cdots : z_{i-1} : 0 : z_{i+1} : \cdots : z_n) : z_j \in \mathbb{C}, j \neq i\}$ which is a $\mathbb{P}^{n-1}_{\mathbb{C}}$. This set is called the *hyperplane at infinity* with respect to U_i.

3.3. Differential forms

This is a very brief description of differential forms. We urge the interested reader to refer to [5].

Consider the canonical basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$ (or $\mathbb{C}^n$). The dual basis $\{dx_1, dx_2, \ldots, dx_n\}$ is defined by

$$dx_i(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Let Ω^* be the real (complex) algebra generated by $dx_1, dx_2, \ldots, dx_n$ subjected to the relations

$$dx_i \wedge dx_i = 0, \quad dx_i \wedge dx_j = -dx_j \wedge dx_i \text{ if } i \neq j.$$

As a real (complex) vector space a basis of Ω^* is given by

$$dx_i (1 \leq i \leq n),\ dx_i \wedge dx_j (i < j),\ dx_i \wedge dx_j \wedge dx_k (i < j < k),\ \ldots,\ dx_1 \wedge \cdots \wedge dx_n$$

Differential forms on $\mathbb{R}^n$ ($\mathbb{C}^n$) are defined by

$$\Omega^*(\mathbb{R}^n) \text{ or } \Omega^*(\mathbb{C}^n) = \{\text{functions}\} \otimes_{\mathbb{R} \text{ or } \mathbb{C}} \Omega^*$$

and, to be more precise, differential forms of class C^k are the elements of

$$\Omega^*_k(\mathbb{R}^n) = C^k(\mathbb{R}^n, \mathbb{R}) \otimes_{\mathbb{R}} \Omega^*(\mathbb{R}^n)$$

and similarly for $\mathbb{C}^n$.

We have a grading of $\Omega^*(\mathbb{R}^n)$ given by

$$\Omega^*(\mathbb{R}^n) = \bigoplus_{p=0}^{n} \Omega^p(\mathbb{R}^n)$$

where $\Omega^p(\mathbb{R}^n)$ consists of the differential forms of degree p or p-forms.

Hence a p-form has an expression

$$\sum_{i_1 < \cdots < i_p} f_{i_1, \ldots, i_p}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

which we note as

$$\sum_I f_I\, dx_I \quad \text{where } I = \{i_1, \ldots, i_n\},\ \ i_1 < \cdots < i_p.$$

Assume the coefficients of differential forms are of class C^k, $k \geq 1$. The *exterior differential* d is the operator

$$d : \Omega^p_k(\mathbb{R}^n) \longrightarrow \Omega^{p+1}_{k-1}(\mathbb{R}^n)$$

given by:

(i) On 0-forms f (functions), $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i$.

(ii) On p-forms $\omega = \sum f_I dx_I$, $d\omega = \sum df_I \wedge dx_I$.

The *wedge product* $\wedge$ of two forms is defined by: if $\omega = \sum_I f_I\, dx_I$ and $\eta = \sum_J\, dx_J$ then

$$\omega \wedge \eta = \sum_{I,J} f_I g_J\, dx_I \wedge dx_J$$

in this order.

As exercises, show that if ω is a p-form and η is a q-form then:

(i) $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$.

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$.

(iii) $d(d\omega) = 0$, that is, $d^2 = 0$.

Suppose now we have a C^∞ map $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. The *pull-back* of a p-form on $\mathbb{R}^n$ by f is a p-form on $\mathbb{R}^m$ defined by

(i) For functions (0-forms) g, $f^*(g) = g \circ f$.

(ii) For p-forms $\omega = \sum_I g_I dx_I$, $f^*(\omega) = \sum_I (g_I \circ f)\, df_I$.

(iii) d commutes with f^*, that is, $d(f^*\omega) = f^*(d\omega)$ (exercise).

A form u is closed if $du = 0$ and exact if $u = dv$ for some form v.

Finally, a cohomological complex $K^\bullet = \bigoplus_{q \in \mathbb{Z}} K^q$ is a collection of modules over a ring, endowed with differentials, that is, linear maps $d^q : K^q \longrightarrow K^{q+1}$ satisfying $d^{q+1} \circ d^q = 0$.

The associated cocycle, coboundary and cohomology modules are defined respectively by

$$Z^q(K^\bullet) = \ker d^q\,, \qquad Z^q(K^\bullet) \subset K^q$$
$$B^q(K^\bullet) = \operatorname{Im} d^{q-1}\,, \qquad B^q(K^\bullet) \subset Z^q(K^\bullet) \subset K^q$$
$$H^q(K^\bullet) = Z^q(K^\bullet)/B^q(K^\bullet)$$

Let $\mathcal{A}^0(M)$ be the $\mathbb{C}$-algebra $C^\infty(M,\mathbb{C})$ and $\mathcal{A}^p(M)$ the $\mathcal{A}^0(M)$-module of C^∞ complex p-forms on M. If M is a complex manifold, the De Rham complex of M is the cohomological complex

$$\mathcal{A}^\bullet_\infty(M) = \bigoplus_{q\geq 0} \mathcal{A}^q_\infty(M)$$

with differential d, the exterior derivative. We denote its cohomology groups by $H^q_{DR}(M,\mathbb{R}) = Z^q(M,\mathbb{R})/B^q(M,\mathbb{R})$.

A C^∞ p-form ω on a complex manifold M is expressed, in local coordinates, as a sum of terms of the types $f_I dx_I$, $g_J dy_J$ and $h_K d(x,y)_K$, where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, $dy_J = dy_{j_1} \wedge dy_{j_2} \wedge \cdots \wedge dy_{j_p}$, $d(x,y)_K$ is a product of p-forms of the types dx_m, dy_n and f_I, g_J, h_K are complex valued functions. Now, $dx_i = (1/2)(dz_i + d\bar{z}_i)$ and $dy_i = (1/2\mathrm{i})(dz_i - d\bar{z}_i)$. Substituting in the terms whose sum is ω, we deduce that a p-form on M can be written as

$$\omega \;=\; \sum k_{i_1,\ldots,i_r,j_1,\ldots,j_s} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

which we abbreviate as $\omega = \sum k_{I,J} dz_I \wedge d\bar{z}_J$. We say that each term of this sum is a p-form of type (r,s), $r+s=p$. The fact that a form is of type (r,s) doesn't depend on the coordinate system since $TM^{\mathbb{C}} = T'M \oplus T''M$ (exercise). Besides, a p-form ω is expressed in a unique way as a sum

$$\omega \;=\; \omega^{(p,0)} + \omega^{(p-1,1)} + \cdots + \omega^{(0,p)}, \tag{33}$$

where $\omega^{(r,s)}$ is of type (r,s).

The decomposition (33) induces a decomposition

$$\mathcal{A}^p(M) \;=\; \mathcal{A}^{(p,0)}(M) \oplus \mathcal{A}^{(p-1,1)}(M) \oplus \cdots \oplus \mathcal{A}^{(0,p)}(M). \tag{34}$$

The exterior differential d complexifies and gives $d : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$, obeying the usual properties. Now, given $f \in \mathcal{A}^0(M)$ locally we have

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

Define

$$\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i \quad \text{and} \quad \overline{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

∂f and $\overline{\partial} f$ do not depend on the coordinate system.

Given $\omega^{(r,s)} = \sum k_{i_1,\dots,i_r,j_1,\dots,j_s} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$ put

$$\partial \omega^{(r,s)} = \sum \partial k_{i_1,\dots,i_r,j_1,\dots,j_s} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$$

a form of type $(r+1, s)$ and

$$\overline{\partial} \omega^{(r,s)} = \sum \overline{\partial} k_{i_1,\dots,i_r,j_1,\dots,j_s} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

of type $(r, s+1)$. This gives

$$d\omega^{(r,s)} = \partial \omega^{(r,s)} + \overline{\partial} \omega^{(r,s)}.$$

As d doesn't depend on the local coordinate system, the same holds for ∂ and $\overline{\partial}$. For an arbitrary p-form $\omega = \sum\limits_{r+s=p} \omega^{(r,s)}$,

$$\partial \omega = \sum_{r+s=p} \partial \omega^{(r,s)} \quad \text{and} \quad \overline{\partial} \omega = \sum_{r+s=p} \overline{\partial} \omega^{(r,s)}.$$

We have $d = \partial + \overline{\partial}$ and

$$\partial(\omega^p \wedge \eta) = \partial \omega^p \wedge \eta + (-1)^p \omega^p \wedge \partial \eta,$$

$$\overline{\partial}(\omega^p \wedge \eta) = \overline{\partial} \omega^p \wedge \eta + (-1)^p \omega^p \wedge \overline{\partial} \eta.$$

Besides,

$$\partial\partial \omega^{(r,s)} + \overline{\partial}\partial \omega^{(r,s)} + \partial\overline{\partial} \omega^{(r,s)} + \overline{\partial}\,\overline{\partial} \omega^{(r,s)} = dd\omega^{(r,s)} = 0.$$

By comparing the types of forms appearing in this equality we get

$$\partial\partial = 0 \ , \ \partial\overline{\partial} + \overline{\partial}\partial = 0 \ , \ \overline{\partial}\,\overline{\partial} = 0.$$

A $(p,0)$-form $\omega^{(p,0)} = \sum f_{i_1,\dots,i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ is *holomorphic* if the coefficients $f_{i_1,\dots,i_p}$ are holomorphic functions. In this case

$$\overline{\partial} \omega = \sum \overline{\partial} f_{i_1,\dots,i_p} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} = 0.$$

Reciprocally $\overline{\partial}\omega^{(p,0)} = 0$ implies that the coefficients of ω are holomorphic functions. Hence, for holomorphic forms we have $\partial\omega = d\omega$.

Lastly, a real manifold M is *orientable* in case it admits an atlas with all transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ with positive jacobian determinant. Suppose M is oriented by such an atlas. If $u(x) = g(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m$ is a continuous m-form on M, where $m = \dim_{\mathbb{R}} M$ and with compact support in a coordinate system, then we define

$$\int_M u = \int_{\mathbb{R}^m} g\, dx_1 \ldots dx_m.$$

This is independent of the coordinate system (orientability). If u has compact support, we extend this definition of $\int_M u$ by means of a partition of unity. A manifold is orientable if, and only if, it admits a nowhere vanishing continuous m-form (exercise).

Now, if $K \subset M$ is a compact set with piecewise C^1 boundary ∂K, it's possible to give an orientation to ∂K in such a way that for any differential form of class C^1 and of degree $m-1$ we have

$$\int_{\partial K} u = \int_K du.$$

This is Stokes formula.

Also, the orientation of a complex manifold of complex dimension n is determined by its volume form, which locally reads:

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \ = \ \left(\frac{\mathrm{i}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

3.3.1. Remarks on Poincaré duality

We recall very briefly Poincaré's duality [5].

If M is a compact complex manifold of dimension n let us denote by $H_q^s(M, \mathbb{Z})$ and $H_s^q(M, \mathbb{Z})$ the q-th *singular homology* group and the q-th *singular cohomology* group of M with integer coefficients, respectively (refer to [10] for the singular theory). Poincaré's duality theorem states that

$$H_s^q(M, \mathbb{Z}) \ \cong \ H_{2n-q}^s(M, \mathbb{Z}). \tag{35}$$

In general, this isomorphism no longer exists if M is not compact. In this case we must bring in the *singular cohomology with compact support*, defined as follows: the compact subsets of M are partially ordered by inclusion ($K \preceq K'$ if and only if $K \subseteq K'$). The relative cohomology groups $H_s^q(M, M \setminus K)$ make up an inductive system indexed by the compact subsets of M, with $H_s^q(M, M \setminus K) \longrightarrow H_s^q(M, M \setminus K')$ induced by inclusion. Take the direct limit and define

$$H_{sc}^q(M, \mathbb{Z}) \;=\; \underset{K \text{ compact}}{\overrightarrow{\text{lím}}} H_s^q(M, M \setminus K). \tag{36}$$

If M is compact we have $H_{sc}^q(M, \mathbb{Z}) = H_s^q(M, \mathbb{Z})$. With this procedure Poincaré's duality now reads:

$$H_{sc}^q(M, \mathbb{Z}) \;\cong\; H_{2n-q}^s(M, \mathbb{Z}). \tag{37}$$

Tensorizing by $\mathbb{C}$ and invoking the Universal Coefficient theorem we obtain

$$H_{sc}^q(M, \mathbb{C}) \;\cong\; H_{2n-q}^s(M, \mathbb{C}). \tag{38}$$

On the other hand we have two de Rham cohomologies, the one with closed forms, $H_{DR}^*(M, \mathbb{C})$, and the one with closed forms with compact support, $H_{c\,DR}^*(M, \mathbb{C})$. They obviously coincide in case M is compact. Under certain conditions on M (existence of a *good cover*) both cohomologies are finite dimensional and Poincaré's duality reads:

$$H_{DR}^q(M, \mathbb{C}) \;\cong\; \Big(H_{c\,DR}^{2n-q}(M, \mathbb{C})\Big)^*. \tag{39}$$

This result is obtained by showing that the bilinear map

$$\begin{array}{ccc} H_{DR}^q(M, \mathbb{C}) \;\times\; H_{c\,DR}^{2n-q}(M, \mathbb{C}) & \longrightarrow & \mathbb{C} \\ (\omega, \eta) & \longmapsto & \displaystyle\int_M \omega \wedge \eta \end{array} \tag{40}$$

is non-degenerate.

We have the *de Rham theorem* ([5]-théorème 17'):

$$H_{sc}^{2n-q}(M, \mathbb{C}) \;\cong\; H_{DR}^q(M, \mathbb{C}), \tag{41}$$

from which it follows that, since the vector space $H_{c\,DR}^{2n-q}(M, \mathbb{C})$ is finite dimensional,

$$H_{sc}^{2n-q}(M, \mathbb{C}) \;\cong\; H_{c\,DR}^{2n-q}(M, \mathbb{C}), \tag{42}$$

where this isomorphism is not natural since choices of bases are involved. From (38) we get

$$H_q^s(M, \mathbb{C}) \;\cong\; H_{c\,DR}^{2n-q}(M, \mathbb{C}) \cong H_{DR}^{2n-q}(M, \mathbb{C}) \tag{43}$$

where the second isomorphism occurs when M is compact.

3.4. Vector bundles

In what follows, by a *topological space* we mean a connected Hausdorff space with countable basis.

Definition 25. *Let X be a topological space. A real vector bundle of rank n over X is a topological space E equiped with a continuous projection $E \xrightarrow{\pi} X$ satisfying:*

(i) $\pi^{-1}(x) := E_x$ (the fiber of E over $x \in X$) is a real vector space of dimension n, $\forall\, x \in X$.

(ii) There exist an open covering of X, $X = \bigcup_{\alpha \in A} U_\alpha$, and homeomorphisms

$$\Theta_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$$

such that $\forall\, \alpha \in A$, if $x \in U_\alpha$ then

$$\Theta_{\alpha\, x} : E_x \longrightarrow \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n \tag{44}$$

is an isomorphism of real vector spaces.

E is called the total space of the bundle, X is called the base and Θ_α are the local trivializations of E.

Definition 26. *A section of $E \xrightarrow{\pi} X$ is a continuous map $s : X \to E$ such that*

$$(\pi \circ s)(x) = x \quad \forall x \in X \tag{45}$$

that is, $s(x) \in E_x$.

Example 27. The trivial bundle over X, $\underline{\mathbb{R}^n}$, is defined by

$$\begin{array}{l} \underline{\mathbb{R}^n} \;=\; X \times \mathbb{R}^n \\ \downarrow \pi \\ X \end{array}$$

where $\pi(x, v) = x$. If $f : X \xrightarrow{C^0} \mathbb{R}^n$, then its graph $s(x) = (x, f(x))$ is a section of $\underline{\mathbb{R}^n}$. Conversely, a section $s : X \to \underline{\mathbb{R}^n}$ defines a function $f : X \xrightarrow{C^0} \mathbb{R}^n$.

Example 28. TM e TM^* are vector bundles. Sections of these bundles are vector fields and differential 1-forms, respectively.

If $s : X \to E$ is a section of E, then

$$\begin{array}{rcl}(\Theta_\alpha \circ s|_{U_\alpha}) : U_\alpha & \longrightarrow & U_\alpha \times \mathbb{R}^n \\ x & \longmapsto & \Theta_\alpha(s(x)) \ = \ (x, S_\alpha(x))\end{array}$$

is the graph of $S_\alpha : U_\alpha \to \mathbb{R}^n$. Hence, a section is locally a function with values in $\mathbb{R}^n$.

Let $E \xrightarrow{\pi} X$ be a vector bundle of rank n, $\{U_\alpha\}$ a trivializing open cover, and $\{\Theta_\alpha\}$ local trivilizations of E. Given $x \in U_\alpha$, the map $\Theta_{\alpha\, x} : E_x \to \mathbb{R}^n$ is the restriction to E_x of the map $\Theta_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^n$. Denoting by $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we define

$$\Theta_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL(n, \mathbb{R}) \text{ by } \Theta_{\alpha\beta}(x) \ = \ \Theta_{\alpha\, x}\Theta^{-1}_{\beta\, x}. \tag{46}$$

Diagramatically:

$$\begin{array}{ccccc} U_{\alpha\beta} \times \mathbb{R}^n & \xleftarrow{\Theta_\beta} & \pi^{-1}(U_{\alpha\beta}) & \xrightarrow{\Theta_\alpha} & U_{\alpha\beta} \times \mathbb{R}^n \\ (x, v) & & \longmapsto & & (x, \Theta_{\alpha\,\beta}(x)v). \end{array}$$

The $\Theta_{\alpha\,\beta}$ are continuous and satisfy the cocycle relation (product in $GL(n, \mathbb{R})$)

$$\Theta_{\alpha\,\beta}\Theta_{\beta\,\gamma}\Theta_{\gamma\,\alpha} \ = \ I \text{ in } U_{\alpha\,\beta\,\gamma} \text{ and } \Theta_{\alpha\,\beta} \ = \ \Theta^{-1}_{\beta\,\alpha}. \tag{47}$$

The $\Theta_{\alpha\,\beta}$ are called the *transition functions* of E. Remark that, if s is a section of E then

$$\Theta_{\alpha\,\beta}\, S_\beta \ = \ S_\alpha. \tag{48}$$

Definition 29. *Let E and F be vector bundles over X. A morphism $\varphi : E \to F$ is a continuous map such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \pi_E \downarrow & & \downarrow \pi_F \\ X & \xrightarrow{Id} & X \end{array}$$

commutes and $\varphi_{|E_x} : E_x \to F_x$ is a linear map $\forall x \in X$. If φ is a bijection and φ^{-1} is a morphism, then φ is called an isomorphism.

Let's consider this definition in more detail. Suppose $\{U_\alpha\}$ is an open covering which trivializes both E and F. Let $\{\Theta_\alpha\}$ and $\{\eta_\alpha\}$ be the trivializations of E and F, respectively. If φ is a morphism from E to F, then φ induces $\varphi_\alpha : U_\alpha \times \mathbb{R}^n \to U_\alpha \times \mathbb{R}^m$ given by

$$\begin{array}{ccc} U_\alpha \times \mathbb{R}^n & \overset{\Theta_\alpha}{\longleftarrow} & \pi_E^{-1}(U_\alpha) \\ \varphi_\alpha \downarrow & & \downarrow \varphi \\ U_\alpha \times \mathbb{R}^m & \overset{\eta_\alpha}{\longleftarrow} & \pi_F^{-1}(U_\alpha) \end{array} \qquad \varphi_\alpha = \eta_\alpha \circ \varphi \circ \Theta_\alpha^{-1}.$$

Now, φ_α is of the form $\varphi_\alpha(x,v) = (x, a_\alpha(x)v)$. Hence $a_\alpha : U_\alpha \overset{C^0}{\longrightarrow} \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ satisfies

$$\eta_{\alpha\,\beta} a_\beta = a_\alpha \Theta_{\alpha\,\beta} \qquad \forall\ \alpha, \beta. \tag{49}$$

Indeed, by the diagram

$$\begin{array}{ccccc} U_{\alpha\,\beta} \times \mathbb{R}^n & \overset{\Theta_\beta}{\longleftarrow} & \pi_E^{-1}(U_{\alpha\,\beta}) & \overset{\Theta_\alpha}{\longrightarrow} & U_{\alpha\,\beta} \times \mathbb{R}^n \\ \varphi_\beta \downarrow & & \downarrow \varphi & & \downarrow \varphi_\alpha \\ U_{\alpha\,\beta} \times \mathbb{R}^m & \overset{\eta_\beta}{\longleftarrow} & \pi_F^{-1}(U_{\alpha\,\beta}) & \overset{\eta_\alpha}{\longrightarrow} & U_{\alpha\,\beta} \times \mathbb{R}^m \end{array}$$

we get

$$\begin{aligned} \varphi_\beta(x,v) &= \eta_\beta \circ \eta_\alpha^{-1} \circ \varphi_\alpha \circ \Theta_\alpha \circ \Theta_\beta^{-1}(x,v) \\ \Longrightarrow\ \eta_\alpha \circ \eta_\beta^{-1} \circ \varphi_\beta(x,v) &= \varphi_\alpha \circ \Theta_\alpha \circ \Theta_\beta^{-1}(x,v) \\ \Longrightarrow\ (x, \eta_{\alpha\,\beta}(x) a_\beta(x) v) &= (x, a_\alpha(x) \Theta_{\alpha\,\beta}(x) v). \end{aligned}$$

Conversely, a family of maps $a_\alpha : U_\alpha \overset{C^0}{\longrightarrow} \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ determines a morphism from E to F provided

$$\eta_{\alpha\,\beta} a_\beta = a_\alpha \Theta_{\alpha\,\beta} \text{ in } U_{\alpha\,\beta}\ ,\ \forall\, \alpha,\ \beta.$$

Remark that E is isomorphic to the trivial bundle $\underline{\mathbb{R}^n}$ if, and only if, there exist $a_\alpha : U_\alpha \overset{C^0}{\longrightarrow} GL(n, \mathbb{R})$ such that $a_\beta = a_\alpha \Theta_{\alpha\,\beta}$ in $U_{\alpha\,\beta}$, $\forall\, \alpha,\ \beta$.

Exercise 11. This exercise gives an alternative description of vector bundles. Let $\{U_\alpha\}$ be an open cover of X. Suppose we are given a family of continuous functions, defined in $U_{\alpha\,\beta}$, $\Theta_{\alpha\,\beta} : U_{\alpha\,\beta} \to GL(n, \mathbb{R})$ and satisfying $\Theta_{\alpha\,\beta}\Theta_{\beta\,\gamma}\Theta_{\gamma\,\alpha} = I$ in $U_{\alpha\,\beta\,\gamma}$ and $\Theta_{\alpha\,\beta} = \Theta_{\beta\,\alpha}^{-1}$ in $U_{\alpha\,\beta}$ (remark that $\Theta_{\alpha\,\alpha} = I$). Set $\mathcal{F} = \coprod_{\alpha \in A} U_\alpha \times \mathbb{R}^n$ (disjoint union with the obviuos topology) and define the following equivalence relation in $\mathcal{F}$:

$$(\alpha, x, u) \sim (\beta, y, v) \Longleftrightarrow x = y\,, \quad \Theta_{\alpha\,\beta}(x)v = u \quad \text{and} \quad U_{\alpha\,\beta} \neq \emptyset.$$

Show that the quotient $\mathcal{F}/\sim$ has the structure of a real vector bundle of rank n over X, unique up to isomorphism, whose transition functions are the $\Theta_{\alpha\,\beta}$. ◁

Usually bundles are constructed from a family of transition functions, as in the above exercise.

In all that was done above, if we change $\mathbb{R}$ by $\mathbb{C}$ we obtain the notion of a complex vector bundle. Besides, if X is a complex manifold and the trivializations are of class C^∞ or are holomorphic, then we will have C^∞ or holomorphic vector bundles.

3.4.1. Tensor products

If E and E' are vector bundles over X, of ranks n and m respectively, the tensor product $E \otimes E'$ is the bundle whose fiber over $x \in X$ is $E_x \otimes E'_x$. The transition functions of $E \otimes E'$ are

$$\Theta_{\alpha\,\beta}(x) \otimes \Theta'_{\alpha\,\beta}(x) : \mathbb{R}^n \otimes \mathbb{R}^m \longrightarrow \mathbb{R}^n \otimes \mathbb{R}^m.$$

In case E and E' have rank 1, then these transition cocycles are simply the product $\Theta_{\alpha\,\beta}\Theta'_{\alpha\,\beta}$.

3.4.2. Subbundles, quotients and determinants

If $E \xrightarrow{\pi} X$ is a vector bundle, a *subbundle* consists of a subset $F \subset E$ such that the projection π and the local trivializations of E endow F with a real vector bundle structure. Given a subbundle F of E, the fibers F_x are subspaces of E_x and the *quotient bundle* E/F is obtained by taking the quotients E_x/F_x. More precisely, let $\{U_\alpha\}$ be a trivializing open cover of E. We have a diagram

$$\begin{array}{ccccc} U_{\alpha\,\beta} \times \mathbb{R}^n & \overset{\Theta_\beta}{\longleftarrow} & \pi^{-1}(U_{\alpha\,\beta}) & \overset{\Theta_\alpha}{\longrightarrow} & U_{\alpha\,\beta} \times \mathbb{R}^n \\ \uparrow & & \uparrow & & \uparrow \\ U_{\alpha\,\beta} \times \mathbb{R}^m & \overset{\eta_\beta}{\longleftarrow} & \pi_{|F}^{-1}(U_{\alpha\,\beta}) & \overset{\eta_\alpha}{\longrightarrow} & U_{\alpha\,\beta} \times \mathbb{R}^m, \end{array}$$

where the vertical arrows are inclusions. Hence we have $(x, v) \mapsto (x, \Theta_{\alpha\beta}(x)v)$ and $(x, v) \mapsto (x, \eta_{\alpha\beta}(x)v)$. Now, $\forall\ v \in \mathbb{R}^m$ these two maps coincide and so $\Theta_{\alpha\beta}(x)_{|\mathbb{R}^m} = \eta_{\alpha\beta}(x)$. We conclude that $\Theta_{\alpha\beta}$ has the expression

$$\Theta_{\alpha\beta}(x) = \begin{pmatrix} \eta_{\alpha\beta}(x) & \rho_{\alpha\beta}(x) \\ 0 & \zeta_{\alpha\beta}(x) \end{pmatrix}.$$

Consider a short exact sequence

$$0 \to \mathbb{R}^m \xrightarrow{T} \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^m \to 0$$

(T linear). This allows us to define the quotient bundle E/F in the following manner: a vector $w \in \mathbb{R}^n$ can be written as $w = v + u$, $v \in \mathbb{R}^m$, $u \in \mathbb{R}^{n-m}$. Then $\Theta_{\alpha\beta}(x)(v, u) = (\eta_{\alpha\beta}(x)v + \rho_{\alpha\beta}(x)u, \zeta_{\alpha\beta}(x)u)$ and, as $\eta_{\alpha\beta}(x)v + \rho_{\alpha\beta}(x)u \in \mathbb{R}^m$, the class of $\Theta_{\alpha\beta}(x)(v, u)$ in the quotient $\mathbb{R}^n/\mathbb{R}^m$ equals the class of $\zeta_{\alpha\beta}(x)u$. The vector bundle E/F is defined by the transition cocycles $\zeta_{\alpha\beta}$.

The *determinant bundle* of E is the bundle $\det E = \bigwedge^n E$, whose fiber is $\bigwedge^n E_x$. Its transition cocycles are $\det \Theta_{\alpha\beta}$, from which it follows

$$\det \Theta_{\alpha\beta} = \det \eta_{\alpha\beta} \otimes \det \zeta_{\alpha\beta},$$

and we conclude

$$\det E \cong \det F \otimes \det E/F.$$

Remark that we've obtained an isomorphism since in the above argument a choice of bases of the spaces involved is implicit.

Exercise 12. If $0 \longrightarrow F \xrightarrow{f} E \xrightarrow{g} G \longrightarrow 0$ is an exact sequence of vector bundles over X, that is, $0 \longrightarrow F_x \longrightarrow E_x \longrightarrow G_x \longrightarrow 0$ is exact $\forall\, x \in X$, then f identifies F with a subbundle of E and g induces an isomorphism between E/F and G.

3.4.3. The Whitney sum

If E and E' are vector bundles over X of ranks n and m respectively, their *direct sum* $E \oplus E'$ is the bundle over X whose fiber over each $x \in X$ is $E_x \oplus E'_x$. Local trivializations Θ_α and Θ'_α of E and E' (relative to the same open cover $\{U_\alpha\}$ of X) induce local trivializations of $E \oplus E'$ by

$$\Theta_\alpha \oplus \Theta'_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \oplus \mathbb{R}^m.$$

Hence, the transition cocycles of $E \oplus E'$ are given by

$$(\Theta_{\alpha\beta} \oplus \Theta'_{\alpha\beta})(x) = \begin{pmatrix} \Theta_{\alpha\beta}(x) & 0 \\ 0 & \Theta'_{\alpha\beta}(x) \end{pmatrix} : \mathbb{R}^n \oplus \mathbb{R}^m \longrightarrow \mathbb{R}^n \oplus \mathbb{R}^m.$$

3.4.4. The dual bundle

Recall that a linear map $f : V \to W$ induces a linear map $f^T : W^* \to V^*$ defined by

$$\begin{array}{rcccl} f^T : & W^* & \longrightarrow & V^* & \\ & \alpha & \longmapsto & f^T(\alpha) : & V \longrightarrow \mathbb{R} \\ & & & & v \longmapsto (\alpha \circ f)(v), \end{array}$$

whose matrix is the transpose of the matrix representing f. If $\Theta_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a local trivialization of E, then

$$(\Theta_\alpha^T)^{-1} : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^*}$$

is, by definition, a local trivialization of the *dual bundle* E^*. The transition cocycles are

$$(\Theta_\alpha^T)^{-1}\Theta_\beta^T = \left((\Theta_\beta^T)^{-1}\Theta_\alpha^T\right)^{-1} = \left((\Theta_\alpha\Theta_\beta^{-1})^T\right)^{-1} = (\Theta_{\alpha\beta}^T)^{-1}.$$

Once again, if E has rank 1, then the transition cocycles of E^* are $\Theta_{\alpha\beta}^{-1}$.

3.4.5. Pull-back

Consider the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ Y & \xrightarrow{f} & X, \end{array}$$

where f is continuous. f induces a bundle over Y, $f^{-1}E$, called the *pull-back* of E via f. As a set, $f^{-1}E$ is the *fibered product* $Y \times_X E \subset Y \times E$ defined by

$$f^{-1}E = \{(y, e) : f(y) = \pi(e)\}.$$

This is the only maximal subset of $Y \times E$ such that the following diagram commutes

$$\begin{array}{ccccc} Y \times E & \supset & f^{-1}E & \xrightarrow{p_2} & E \\ & & p_1 \downarrow & & \downarrow \pi. \\ & & Y & \xrightarrow{f} & X \end{array}$$

If E is trivial, $E = X \times \mathbb{C}^n$, then

$$f^{-1}E \;=\; \{(y,(x,v)) : x = f(y)\} \cong \{(y,v) : y \in Y\,,\, v \in \mathbb{C}^n\} \;=\; Y \times \mathbb{C}^n.$$

Hence, by using the trivializations of E we deduce that the fiber of $f^{-1}E$ over y is isomorphic to $E_{f(y)}$. Besides, if we have a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$, then $(f \circ g)^{-1}E \cong g^{-1}f^{-1}E$.

Exercise 13. Determine the transition functions of $f^{-1}E$.

3.5. Some examples of holomorphic vector bundles

3.5.1. $[V]$

Let M be a compact complex manifold and $V \subset M$ an analytic set of codimension 1. Cover M by open sets U_i such that V is defined in U_i by ${f_i}^{-1}(0)$ ($f_i : U_i \to \mathbb{C}$, holomorphic and reduced). In U_{ij} we have the relation $f_i = \varphi_{ij} f_j$, where φ_{ij} is holomorphic and vanishes nowhere. The rank 1 vector bundle $[V]$ is defined by the transition functions $\varphi_{ij} = f_i/f_j$ (recall Exercise 6). Remark that $[V]$ admits a holomorphic section s_V defined by V. In fact, on the trivializing open set U_i we have

$$\begin{array}{ccc} U_i \times \mathbb{C} & \xleftarrow{\varphi_i} & [V]_{|U_i} \\ & p_1 \searrow \qquad \swarrow \pi & \\ & U_i & \end{array}$$

and the section $s_V : U_i \to [V]_{|U_i}$ is defined by the graph of the function f_i, $s_{V_{|U_i}}(z) = \varphi_i^{-1}(z, f_i(z))$. Moreover, the zeros of s_V define V. Suppose now that, in U_i, V is also defined by $g_i = 0$. Then f_i/g_i does not vanish at any point in U_i and

$$\beta_{ij} \;=\; \frac{g_i}{g_j} \;=\; \frac{f_i}{f_j}\frac{\frac{f_j}{g_j}}{\frac{f_i}{g_i}} \;=\; \varphi_{ij}\frac{\frac{f_j}{g_j}}{\frac{f_i}{g_i}},$$

that is (see (49)),

$$\frac{f_i}{g_i}\beta_{ij} \;=\; \varphi_{ij}\frac{f_j}{g_j}$$

and we conclude that the bundle defined by the φ_{ij}'s is isomorphic to the bundle defined by the β_{ij}'s. Hence, the isomorphism class of the bundle $[V]$ is associated to V.

3.5.2. The tautological bundle $\mathcal{O}^*$

Consider the trivial bundle $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \xrightarrow{\pi} \mathbb{P}^n_{\mathbb{C}}$. The *tautological* or *universal* bundle is the rank 1 subbundle consisting of the pairs $([w], z) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$ such that z belongs to the straight line defined by $[w]$ (hence the reason for the term tautological):

$$\mathcal{O}^* \;=\; \{([w], z) : \exists\, t \in \mathbb{C} \text{ such that } z = tw\}.$$

Recalling the definition of $\mathbb{P}^n_{\mathbb{C}}$ in (3.2.1) we have, in the open set U_i,

$$\mathcal{O}^*_{|U_i} \;=\; \{((z_0 : \cdots : z_n), t(z_0, \ldots, z_n)) \;;\; t \in \mathbb{C}\}\,.$$

The transition cocycles are defined by

$$\begin{array}{ccccc} U_{ij} \times \mathbb{C} & \xleftarrow{\Theta_j} & \mathcal{O}^*_{|U_{ij}} & \xrightarrow{\Theta_i} & U_{ij} \times \mathbb{C} \\ ([z], t) & & \longmapsto & & (x, \Theta_{ij}([z])t). \end{array}$$

and, in U_{ij},

$$\left(\frac{z_0}{z_j}, \ldots, \frac{z_i}{z_j}, \ldots, 1, \ldots, \frac{z_n}{z_j}\right) \;=\; \left(\frac{z_i}{z_j}\right)\left(\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_j}{z_i}, \ldots, \frac{z_n}{z_i}\right)$$

and hence,

$$\Theta_{ij}([z]) \;=\; \frac{z_i}{z_j} \quad \text{in } U_{ij}.$$

3.5.3. The hyperplane bundle $\mathcal{O}$

Let H be a hyperplane in $\mathbb{P}^n_{\mathbb{C}}$, that is, $H = \{P(z) = 0\}$, where $P : \mathbb{C}^{n+1} \to \mathbb{C}$ is a polynomial of degree 1, $P(0) = 0$. By a linear change of coordinates we may assume $H = \{z_0 = 0\}$. In U_i, $i \neq 0$, H is defined by $f_i = z_0/z_i = 0$. The rank 1 bundle $\mathcal{O}$, which represents the isomorphism class of the bundles of the form $[H]$ (see (3.5.1)), is defined by the transition functions

$$\varphi_{ij} \;=\; \frac{f_i}{f_j} \;=\; \frac{\dfrac{z_0}{z_i}}{\dfrac{z_0}{z_j}} \;=\; \frac{z_j}{z_i} \quad \text{in } U_{ij}.$$

Exercise 14. Show that any two hyperplanes define isomorphic bundles.

Notice that, by 3.4.4, $\mathcal{O}$ is the dual of $\mathcal{O}^*$ since $\varphi_{ij} = \Theta_{ij}^{-1}$. Now, given $d \in \mathbb{Z}$ the bundle $\mathcal{O}(d)$ is defined by

$$\mathcal{O}(d) = \begin{cases} \mathcal{O}^{\otimes d} = \underbrace{\mathcal{O} \otimes \cdots \otimes \mathcal{O}}_{d \text{ times}}, & \text{if } d \geq 0, \\ \\ \mathcal{O}^{*\otimes -d} = \underbrace{\mathcal{O}^* \otimes \cdots \otimes \mathcal{O}^*}_{-d \text{ times}}, & \text{if } d \leq 0. \end{cases}$$

3.5.4. The canonical bundle K_M

If M is a complex manifold, the *canonical bundle* K_M of M is defined by $K_M = (\det T'M)^*$. In case $M = \mathbb{P}^n_{\mathbb{C}}$, since the transition cocycles of $\det T'\mathbb{P}^n_{\mathbb{C}}$ are

$$\det \Theta_{ij} = (-1)^{i+j} \left(\frac{z_j}{z_i}\right)^{n+1},$$

we have

$$(-1)^i \det \Theta_{ij} = \left(\frac{z_j}{z_i}\right)^{n+1} (-1)^j$$

and, by (49), $\det T'\mathbb{P}^n_{\mathbb{C}}$ is isomorphic to the bundle whose transition cocycles are $(z_j/z_i)^{n+1}$, that is, $\det T'\mathbb{P}^n_{\mathbb{C}} \cong \mathcal{O}(n+1)$. Hence,

$$K_{\mathbb{P}^n_{\mathbb{C}}} \cong \mathcal{O}(n+1)^* \cong \mathcal{O}(-n-1).$$

3.6. The canonical bundle K_V

Suppose $V \hookrightarrow M$ is a compact submanifold of the complex manifold M. The *normal bundle* N_V of V in M is, by definition, the quotient

$$N_V = \frac{T'M_{|V}}{T'V}.$$

Hence, we have the short exact sequence

$$0 \longrightarrow T'V \longrightarrow T'M_{|V} \longrightarrow N_V \longrightarrow 0$$

and, by 3.4.2 , $(\det T'M)_{|V} \cong \det T'V \otimes \det N_V$. But these are rank 1 vector bundles and therefore

$$K_V \cong K_{M_{|V}} \otimes \det N_V.$$

Suppose now that V has codimension 1. Then this last formula reads $K_V \cong K_{M_{|V}} \otimes N_V$. Take a trivializing open cover $\{U_i\}$ common to all these bundles. Then, in U_i, V is defined by $f_i = 0$ (since it's a submanifold of dimension $n-1$) and $[V]$ is determined by the transition cocycles $\varphi_{ij} = f_i/f_j$ (see (3.5.1)).

Now, in U_{ij}, $f_i = \varphi_{ij} f_j$, which gives $df_i = f_j \, d\varphi_{ij} + \varphi_{ij} \, df_j$ and, along V, $df_i = \varphi_{ij} df_j$. On the other hand, in $U_i \cap V$, $T'V$ is defined as the kernel of df_i and hence df_i defines a nowhere vanishing holomorphic section of the dual bundle N_V^*, since V is a submanifold.

Let $\zeta_i : \pi^{-1}(U_i \cap V) \to U_i \cap V \times \mathbb{C}$ be a local trivialization of N_V^* and put $S_i = \zeta_i df_i$. Then, $\zeta_i^{-1} S_i = \varphi_{ij} \zeta_j^{-1} S_j$, that is, $S_i = \zeta_i \zeta_j^{-1} \varphi_{ij} S_j = \zeta_{ij} \varphi_{ij} S_j$, and by (48) we have that the df_i's define a global nowhere vanishing holomorphic section of $N_V^* \otimes [V]_{|V}$. This says that $N_V^* \otimes [V]_{|V}$ is isomorphic to the trivial bundle and hence

$$N_V \cong [V]_{|V}.$$

From this we deduce the *adjunction formula*

$$K_V \cong K_{M_{|V}} \otimes [V]_{|V}. \tag{50}$$

Exercise 15. Let E be a holomorphic vector bundle of rank n over M. If E admits n linearly independent holomorphic sections, then E is holomorphically isomorphic to the trivial bundle $\underline{\mathbb{C}^n}$.

4. Currents

Let $\mathcal{A}^\ell_{\infty,c}(\mathbb{R}^n) = \mathcal{A}^\ell_c(\mathbb{R}^n)$ be the space of C^∞ ℓ-forms on $\mathbb{R}^n$ with compact support. The topological dual of $\mathcal{A}^{n-q}_c(\mathbb{R}^n)$ is the space of *currents of degree* q, denoted $\mathcal{D}^q(\mathbb{R}^n)$. This means that $\mathcal{D}^q(\mathbb{R}^n)$ is the space of continuous linear forms T on $\mathcal{A}^{n-q}_c(\mathbb{R}^n)$.

For a good account on currents refer to [4].

Remark 30. *The topological background necessary to treat currents is not elementary and we will only say that the topology involved is based on seminorms. The topology in* $\mathcal{A}^\ell_{\infty,c}(\mathbb{R}^n)$ *doesn't make it a complete space.*

We begin by giving some examples.

Example 31. *Let* $\mathsf{L}^q_{loc}(\mathbb{R}^n)$ *be the space of* q*-forms* $u(x) = \sum_I u_I(x) dx_I$ *whose coefficients* $u_I(x)$ *are locally integrable. Then*

$$T_u(\phi) = \int_{\mathbb{R}^n} u \wedge \phi \,, \qquad \phi \in \mathcal{A}^{n-q}_c(\mathbb{R}^n) \tag{51}$$

is the degree q current associated to u. The assignment $u \longmapsto T_u$ is injective (compare with Proposition 13) and we will identify the current T_u with the form u.

Example 32. *Let Γ be a piecewise smooth oriented $(n-q)$-chain in $\mathbb{R}^n$. Note that Γ could be a closed oriented $(n-q)$-submanifold, with boundary $\partial\Gamma$. Then*

$$T_\Gamma(\phi) = \int_\Gamma \phi\,, \qquad \phi \in \mathcal{A}_c^{n-q}(\mathbb{R}^n) \tag{52}$$

is the current of integration over Γ.

This illustrates the concept of support: the *support of a current T*, *supp*(T), is the smallest closed set S such that $T(\phi) = 0$ for all $\phi \in \mathcal{A}_c^{n-q}(\mathbb{R}^n \setminus S)$. In the above case *supp*$(T_\Gamma) = \Gamma$.

The exterior derivative induces an operator

$$d : \mathcal{D}^q(\mathbb{R}^n) \longrightarrow \mathcal{D}^{q+1}(\mathbb{R}^n)$$

which is, by definition:

$$(dT)(\phi) = (-1)^{q+1} T(d\phi)\,, \qquad \phi \in \mathcal{A}_c^{n-q-1}(\mathbb{R}^n) \tag{53}$$

and it satisfies $d^2 = 0$ (exercise). Compare with (20). This is the beginning of the *residue theory*.

A current is *closed* in case $dT = 0$.

In the case of Example 31,

$$(dT_u)(\phi) = (-1)^{q+1} \int_{\mathbb{R}^n} u \wedge d\phi = -\int_{\mathbb{R}^n} \underbrace{d(u \wedge \phi)}_{=0} + \int_{\mathbb{R}^n} du \wedge \phi = T_{du}(\phi). \tag{54}$$

In the case of Example 32, by Stokes (with a proper choice of orientations),

$$(dT_\Gamma)(\phi) = (-1)^{q+1} \int_\Gamma d\phi = (-1)^{q+1} \int_{\partial\Gamma} \phi = (-1)^{q+1} T_{\partial\Gamma}(\phi). \tag{55}$$

Let $\omega \in \mathsf{L}^q_{loc}(\mathbb{R}^n)$ be C^∞ outside a closed set S. Suppose that $d\omega$ on $\mathbb{R}^n \setminus S$ extends to a locally integrable form on $\mathbb{R}^n$.

Definition 33. *The residue is the current defined by*

$$Res(\omega) = dT_\omega - T_{d\omega}. \tag{56}$$

We have *supp* $Res(\omega) \subset S$. This is known as the *localization principle*.

In $\mathbb{C}$, consider the Cauchy kernel

$$\kappa = \frac{1}{2\pi \mathrm{i}} \frac{dz}{z}.$$

Exercise 16. Show that the function $z \longmapsto \dfrac{1}{z}$ is locally integrable.

The following is a very useful result (see [6]):

Proposition 34. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a C^∞ function and let γ be a simple closed curve in $\mathbb{C}$ whose interior is the open set D. Then, for $w \in D$*

$$f(w) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z-w}\, dz + \frac{1}{2\pi \mathrm{i}} \int\int_D \frac{\partial f}{\partial \bar z}(z) \, \frac{dz \wedge d\bar z}{z-w}.$$

Remark that, in case f is holomorphic this reduces to Cauchy's integral formula since $\dfrac{\partial f}{\partial \bar z}(z) = 0$.

Returning to the Cauchy kernel we have $\kappa \in \mathsf{L}^{(1,0)}_{loc}(\mathbb{C})$ and is C^∞ on $\mathbb{C}\backslash\{0\}$, $d\kappa = \overline{\partial}\kappa = 0$ on $\mathbb{C} \setminus \{0\}$ and by Proposition 34, for $\phi \in C^\infty_c(\mathbb{C})$

$$\phi(0) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}^2} \frac{\partial \phi(z)}{\partial \bar z} \frac{dz \wedge d\bar z}{z}.$$

Hence $T_{d\kappa} = 0$ and $dT_\kappa = \overline{\partial} T_\kappa$. But this reads $(\overline{\partial} T_\kappa)(\phi) = \phi(0) = \delta_0(\phi)$ and

$$Res(\kappa) = \delta_0$$

the Dirac function.

This can be generalized to $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by means of the Bochner-Martinelli kernel. We start with a kernel in $\mathbb{C}^n \times \mathbb{C}^n$, which is the complex analogue of the *Newtonian potential* in $\mathbb{R}^n \times \mathbb{R}^n$:

$$G(w,z) = \begin{cases} -\dfrac{1}{2\pi} \log |w-z|^2, & \text{for } n = 1 \\[2ex] \dfrac{(n-2)!}{2\pi^n} |w-z|^{2-2n}, & \text{for } n \geq 2. \end{cases}$$

In what follows, w will denote the variable of integration, z will be a parameter and we let

$$\alpha_{2n-1} = \frac{2\pi^n}{(n-1)!} \quad \text{and} \quad \Lambda = |w-z|^2.$$

Notice that, since the area of the sphere $S_R^{2n-1} \subset \mathbb{C}^n$ of radius R is $\alpha_{2n-1}\, R^{2n-1}$, α_{2n-1} is just the area of the unit sphere S_1^{2n-1}.

The Bochner-Martinelli kernel (for functions) is the double form

$$K(w,z) = - * \partial_w G(w,z)$$

of type $(n, n-1)$ in w and type $(0,0)$ in z.

$K(w,z)$ is represented by the form

$$K = \frac{(n-1)!}{(2\pi \mathrm{i})^n |w-z|^{2n}} \sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right).$$

Set $n = 1$ to get the Cauchy kernel

$$\kappa = \frac{1}{2\pi \mathrm{i}} \frac{dw}{w-z}.$$

We have $\overline{\partial}_w K(w,z) = 0$ on $\mathbb{C}^n \times \mathbb{C}^n \setminus \{w = z\}$. K normalizes the area of spheres, more precisely: let $B_\epsilon(z)$ denote the euclidean ball centered at z and with radius ϵ. Then,

$$\int_{\partial B_\epsilon(z)} K(w,z) = 1$$

for all $z \in \mathbb{C}^n$ and for all $\epsilon > 0$.

Finally we have the Bochner-Martinelli integral formula

Theorem 35. *Let $U \subset \mathbb{C}^n$ be a limited domain whose boundary ∂U is a smooth manifold. Suppose $f : \bar{U} \to \mathbb{C}$ is continuous and f is holomorphic in U. Then,*

$$\int_{\partial U} f(w)\, K(w,z) = \begin{cases} f(z), & \text{for } z \in U \\ 0, & \text{for } z \notin U. \end{cases}$$

Proceeding verbatim as we did in the case of the Cauchy kernel in $\mathbb{C}$, we have that

$$\overline{\partial}_w T_K = \delta_z$$

and

$$Res(K) = \delta_z.$$

A current $T \in \mathcal{D}^q(\mathbb{R}^n)$ may be expressed as a differential form whose coefficients T_I are distributions. Such a current can be written in a unique way as

$$T = \sum_{|I|=q} T_I \, dx_I \tag{57}$$

where T_I is a current of degree 0. This is done by introducing the *wedge product*. If $T \in \mathcal{D}^q(\mathbb{R}^n)$ and $\omega \in \mathcal{A}_c^r(\mathbb{R}^n)$ then $T \wedge \omega \in \mathcal{D}^{q+r}(\mathbb{R}^n)$ is defined by

$$(T \wedge \omega)(\eta) = T(\omega \wedge \eta)\,, \qquad \eta \in \mathcal{A}_c^{n-q-r}(\mathbb{R}^n). \tag{58}$$

It follows that (exercise)

$$d(T \wedge \omega) = dT \wedge \omega + (-1)^q T \wedge d\omega.$$

The decomposition (57) is obtained in the following way:

Let $I = (i_1, \dots, i_q)$, $i_1 < \cdots < i_q$, let $(I, \complement I)$ be the permutation $(1, \dots, n) \longleftrightarrow (I, \complement I)$ and $\sigma_I = \pm 1$ be the sign of this permutation. Given $\phi \in \mathcal{D}(\mathbb{R}^n)$ identify it with the n-form $\phi \, dx_1 \wedge \cdots \wedge dx_n \in \mathcal{A}_c^n(\mathbb{R}^n)$ and put

$$T_I(\phi) = T_I(\phi \, dx_1 \wedge \cdots \wedge dx_n) = \sigma_I \, T(\phi \, dx_{\complement I}).$$

Doing this for all I such that $|I| = q$ we obtain the decomposition (57). In particular, recalling Definition 16, if the T_I are distributions of order 0 then they define Radon measures and in this case the current T is identified with a differential form whose coefficients are measures.

If M is a complex manifold, the currents $\mathcal{D}^{(p,p)}(M)$ of type (p, p) are the continuous linear forms on $\mathcal{A}_c^{n-p,n-p}(M)$. A (p, p)-current is real if $T = \overline{T}$, that is, $\overline{T(\phi)} = T(\overline{\phi})$ for all $\phi \in \mathcal{A}_c^{n-p,n-p}(M)$. A real current is positive if

$$\mathrm{i}^{p(p-1)/2} T(\eta \wedge \overline{\eta}) \geq 0, \qquad \eta \in \mathcal{A}_c^{n-p,0}(M). \tag{59}$$

The positivity of T implies that it has order 0 in the sense of distributions and hence defines a *positive measure*, by Definition 18 and Theorem 19.

An important example is given by Example 32: if $Z \subset M$ is a codimension p analytic subvariety and Z_{reg} is the set of smooth points of Z, then the map

$$T_Z(\phi) = \int_{Z_{reg}} \phi, \qquad \phi \in A_c^{n-p,n-p}(M)$$

defines a closed positive current, which is the fundamental class of Z.

A C^∞ $(1,1)$-form

$$\omega = \frac{\mathrm{i}}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j$$

is real if $\overline{h_{ij}} = h_{ji}$, positive if the matrix h_{ij} is positive definite and closed when the associated hermitian metric $ds^2 = \sum_{i,j} h_{ij} dz_i d\bar{z}_j$ is Kähler.

A real function $\phi \in L^1_{loc}(M)$ is plurisubharmonic in case $\mathrm{i}\partial\overline{\partial}\phi$ is a positive $(1,1)$-current (derivatives are in the sense of distributional derivatives). There is the $\partial\overline{\partial}$-Poincaré lemma:

Let T be a closed, positive $(1,1)$-current. Then, locally,

$$T = \mathrm{i}\partial\overline{\partial}\phi$$

for a real plurisubharmonic function ϕ, uniquely determined up to addition of the real part of a holomorphic function.

5. An application involving holomorphic foliations

5.1. One-dimensional foliations on $\mathbb{P}^n_{\mathbb{C}}$

The content of this section stems from M. Brunella's article [1]. For a broad account on holomorphic foliations see [3].

Let us now consider foliations of dimension one on complex projective spaces. Recall the hyperplane bundle $\mathcal{O}$. A computation shows that any bundle map $\mathcal{O}(m) \to T\mathbb{P}^n_{\mathbb{C}}$ is identically zero for $m \geq 2$ and hence one-dimensional (singular) holomorphic foliations $\mathcal{F}$ of $\mathbb{P}^n_{\mathbb{C}}$ are given by morphisms

$$\Psi : \mathcal{O}(1-d) \longrightarrow T\mathbb{P}^n_{\mathbb{C}}$$

where $d \geq 0$(here, $T\mathbb{P}^n_{\mathbb{C}}$ is the holomorphic tangent bundle of $\mathbb{P}^n_{\mathbb{C}}$). The integer d is called the *degree* of $\mathcal{F}$. Such a foliation is defined locally by a polynomial vector field whose expression is as follows: in affine coordinates $(z_1, \ldots, z_n)$ $\mathcal{F}$ is given by the orbits of a polynomial vector field of degree $d+1$ or d, of the following form:

$$\xi = g\,R + \sum_{j=0}^{d} Y_j, \tag{60}$$

where $R = \sum_{i=1}^{n} z_i \dfrac{\partial}{\partial z_i}$ is the radial vector field, $g \in \mathbb{C}[z_1, \ldots, z_n]$ is homogeneous of degree d and $Y_j = \sum_{i=1}^{n} Y_{ji} \dfrac{\partial}{\partial z_i}$ with $Y_{ji} \in \mathbb{C}[z_1, \ldots, z_n]$ homogeneous of degree j. ξ is a rational vector field on $\mathbb{P}^n_{\mathbb{C}}$ with a pole of order $d-1$ along the hyperplane at infinity $\mathbb{P}^{n-1}_{\mathbb{C}}$. Therefore, in order to cancel the pole we tensorize $T\mathbb{P}^n_{\mathbb{C}}$ by $\mathcal{O}(d-1)$ and obtain a holomorphic section (which we still call ξ) $\xi : \mathbb{P}^n_{\mathbb{C}} \to T\mathbb{P}^n_{\mathbb{C}} \otimes \mathcal{O}(d-1)$. If $g \not\equiv 0$ then, the hyperplane at infinity $\mathbb{P}^{n-1}_{\mathbb{C}}$ is not $\mathcal{F}$-invariant and $\{z \in \mathbb{P}^{n-1}_{\mathbb{C}} : g(z) = 0\}$ is precisely the variety of tangencies of the leaves of $\mathcal{F}$ with $\mathbb{P}^{n-1}_{\mathbb{C}}$ (this explains why d is called the degree of $\mathcal{F}$: d is the degree of the variety of tangencies of $\mathcal{F}$ with a generic hyperplane), whereas in case $g \equiv 0$ and Y_d is not of the form $h\,R$, with h homogeneous of degree $d-1$, the hyperplane at infinity $\mathbb{P}^{n-1}_{\mathbb{C}}$ is $\mathcal{F}$-invariant and the restriction $\mathcal{F}_{|\mathbb{P}^{n-1}_{\mathbb{C}}}$ gives a foliation of $\mathbb{P}^{n-1}_{\mathbb{C}}$ of degree d. Remark that the representation $\xi = g\,R + \sum_{j=0}^{d} Y_j$ is unique up to multiplication by a non-zero complex number since we are only interested in the direction defined by ξ.

The line bundle $\mathcal{O}(1-d)$ is called the *tangent bundle* of $\mathcal{F}$ and noted $T\mathcal{F}$. Its dual $\mathcal{O}(d-1)$ is the *cotangent bundle* $T^*\mathcal{F}$.

The *singular set* of $\mathcal{F}$, $S(\mathcal{F})$, is the set of zeros of the vector field ξ of (60). We will assume that $S(\mathcal{F})$ consists only of isolated points. Since $\mathbb{P}^n_{\mathbb{C}}$ is compact this implies that $S(\mathcal{F})$ is a finite set of points. Moreover we will suppose that all the singularities of $\mathcal{F}$ are *hyperbolic*. This means the following:

Definition 36. *Let X be a holomorphic vector field defined on a neighborhood of a point $p \in \mathbb{C}^n$. Suppose $X(p) = 0$ and the derivative $DX(p)$ satisfies* $\det DX(p) \neq 0$. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $DX(p)$. p is a hyperbolic singularity if $\lambda_i \notin \mathbb{R}.\lambda_j$ for all $i \neq j$.*

These singularities have interesting properties, like the one in the proposition below, but first we need the concept of invariant branch. If X is a holomorphic vector field defined around a point p with $X(p) = 0$, an *invariant branch (separatrix)* for X at p is a non-constant curve Γ passing through p such that, for each $q \in \Gamma \setminus \{p\}$ we have $X(q) \in T_q\Gamma (T_q\Gamma$ denotes the holomorphic tangent bundle of Γ).

Proposition 37. *Let X and p be as above. Suppose p is a hyperbolic singularity of X and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $DX(p)$. Then X has exactly n invariant branches through p, say $\Gamma_1, \ldots, \Gamma_n$, such that:*

(i) Γ_i is smooth at p, $1 \le i \le n$.

(ii) For each eigendirection v_j of $DX(p)$, there is exactly one $i \in \{1, \ldots, n\}$ such that Γ_i is tangent to v_j at p.

(iii) If Γ is an invariant branch of X at p then, as germs at p, $\Gamma = \Gamma_i$ for some i.

That is, through a hyperbolic singularity pass n smooth separatrices. For a proof see [8]. We remark that this proposition holds with the weaker hypothesis $\lambda_i \notin \mathbb{R}^+.\lambda_j$ for all $i \neq j$.

We saw in (59) the concept of positive current. Now, suppose we have a one-dimensional foliation $\mathcal{F}$ on $\mathbb{P}^n_{\mathbb{C}}$ and let T be a closed positive current of type $(1,1)$ defined on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$. T is said to be *invariant* by $\mathcal{F}$ if $T(\varpi) = 0$ for every 2-form ϖ which vanishes on the leaves of $\mathcal{F}$, that is to say that the value $T(\varpi)$ depends only on the restriction of ϖ to the leaves of $\mathcal{F}$. More explicitly, on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$ we may choose coordinates $(z_1, \ldots, z_n)$ in such a way that $\mathcal{F}$ is induced by the vector field $\dfrac{\partial}{\partial z_1}$. Put

$$\alpha_j = dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n\,, \quad 1 \leq j \leq n.$$

In these coordinates T can be written locally as

$$T = \sum_{j,k=1}^{n} h_{jk}\, \mathrm{i}\, \alpha_j \wedge \bar{\alpha}_k$$

where h_{jk} are complex measures. Since T is invariant by $\mathcal{F}$ we have $T \wedge d_j = T \wedge d\bar{z}_j = 0$ for all $j \neq 1$, which gives $h_{jk} = 0$ whenever $(j,k) \neq (1,1)$, so that

$$T = h_{11}\, \mathrm{i}\, \alpha_1 \wedge \bar{\alpha}_1.$$

Since $dT = 0$ we have that the distributional derivatives of h_{11} with respect to z_1 and $\bar{z}_1$ are zero. Hence, T does not depend on z_1 and can be projected to give a positive measure on the local transversal section $\{z_1 = 0\}$. Repeating this procedure on each foliated chart on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$ we associate to T a measure which is transverse to $\mathcal{F}$ and invariant by the holonomy.

Reciprocally, given such a measure it is possible to obtain on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$ a closed positive current T which is invariant by $\mathcal{F}$. This is a consequence of a nontrivial result by D. Sullivan [11] (Caution: this is not an easy reading paper) which states that there is a natural bijective correspondence between invariant measures for $\mathcal{F}_{|\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})}$ and invariant positive currents of type $(1,1)$.

Now $S(\mathcal{F})$ is, by hypothesis, a finite set of points. In this case the current T, defined on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$ admits a unique extension to all of $\mathbb{P}^n_{\mathbb{C}}$, which we still denote by T.

5.2. The residue

Let $p \in S(\mathcal{F})$ and U be an open neighborhood of p. The foliation $\mathcal{F}$ is defined in U by a holomorphic vector field

$$v = X_1 \frac{\partial}{\partial z_1} + \cdots + X_n \frac{\partial}{\partial z_n}$$

having p as the only zero in U. Choose a holomorphic n-form Ω on U without zeros, for instance $\Omega = dz_1 \wedge \cdots \wedge dz_n$ and contract Ω by v that is, look at $(n-1)$-form $\perp_v \Omega$, which is given by

$$\perp_v \Omega = \sum_{i=1}^{n} (-1)^{i-1} X_i \, dz_1 \wedge \ldots \widehat{dz_i} \cdots \wedge dz_n \tag{61}$$

This $(n-1)$-form has a geometric interpretation: on $U \setminus \{p\}$ the kernel of $i_v \Omega$, at each point, is a subspace of dimension $n-1$ which is a complement of the line generated by v. Hence $\perp_v \Omega$ spans the determinant of the conormal bundle of $\mathcal{F}$, $\det N^* \mathcal{F}$.

Consider now the C^∞ $(1,0)$-form β on $U \setminus \{p\}$ defined by

$$\beta = \left(\frac{\operatorname{div}(v)}{\sum_{i=1}^{n} |X_i|^2} \right) \sum_{i=1}^{n} \overline{X}_i \, dz_i \tag{62}$$

where div denotes the divergence of the vector field v, $\operatorname{div}(v) = \sum_{i=1}^{n} \frac{\partial X_i}{\partial z_i}$. The β satisfies (exercise)

$$d \perp_v \Omega = \beta \wedge \perp_v \Omega. \tag{63}$$

Remark that $\beta . v = \operatorname{div}(v) = \operatorname{tr} Dv$ (the trace of the derivative of v). This tells us that β, when restricted to the leaves of $\mathcal{F}$, induces a section of the cotangent bundle $T^*\mathcal{F}$ which, since $\mathcal{F}$ has dimension one, is the same as the canonical bundle $K\mathcal{F}$. It follows from this that $\beta_{|\mathcal{F}}$ is holomorphic and hence holomorphically extendable at p (a fact we will not prove).

Let φ be a test function on U, which is zero on a neighborhood of p and equal to 1 outside a compact subset of U. The C^∞ 2-form $d\varphi \wedge \beta$ is defined in U and has compact support. Define

$$Res(\mathcal{F}, T, p) = \frac{1}{2\pi \mathrm{i}} T(d\varphi \wedge \beta). \tag{64}$$

This is an index, or residue, associated to $\beta_{|\mathcal{F}}$ at p relative to the current T. It is well-defined but we will simply assume this as a fact. The main property of this residue is given in the following

Proposition 38. *Let $\mathcal{F}$ be a holomorphic foliation of dimension one on $\mathbb{P}^n_{\mathbb{C}}$ with finite singular set and let T be a closed positive current of bidimension $(1,1)$ on $\mathbb{P}^n_{\mathbb{C}}$ which is invariant by $\mathcal{F}$. Then*

$$c_1(\det N^*\mathcal{F}).[T] = \sum_{p\in S(\mathcal{F})\cap supp(T)} Res(\mathcal{F},T,p). \tag{65}$$

Proposition 38 is a result of Poincaré-Hopf type. The right hand side of equation (65) is a sum of local indices of the current associated to singular points of the foliation, whereas the left hand side is a number, of global character, associated to $\mathcal{F}$ and T. The term $c_1(\det N^*\mathcal{F})$ is the first Chern class of the line bundle $\det N^*\mathcal{F}$ and it is given by the trace of a curvature matrix (note that the form β satisfies $\beta.v = \mathrm{tr}Dv$). It is represented by a closed C^∞ 2-form and so $c_1(\det N^*\mathcal{F}) \in H^2_{DR}(\mathbb{P}^n_{\mathbb{C}},\mathbb{R})$. $[T] \in H_2(\mathbb{P}^n_{\mathbb{C}},\mathbb{R})$ is the homology class of T and

$$c_1(\det N^*\mathcal{F}).[T] = T(\Theta)$$

where Θ is a convenient compactly supported closed 2-form representing $c_1(\det N^*\mathcal{F})$. The proof is given in [1].

5.3. The theorem

We now bring in the hyperbolic singularities in order to explain a result of M. Brunella [1]. We denote by $Fol(d,n)$ the space of holomorphic foliations on $\mathbb{P}^n_{\mathbb{C}}$ of degree d. This space lies inside a projective space since its points are represented by vector fields ξ as in (60), modulo multiplication by complex numbers, (the coefficients of the vector field are homogeneous coordinates) hence it has a natural topology.

Theorem 39. *Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathcal{U} \subset Fol(d,n)$ such that any $\mathcal{F} \in \mathcal{U}$ has no invariant measure.*

Let's give a very rough idea of how this result is proven. The set $\mathcal{U} \subset Fol(d,n)$ was constructed in [8] and it has the following properties:

(i) if $\mathcal{F} \in \mathcal{U}$ then all singularities of $\mathcal{F}$ are hyperbolic.

(ii) if $\mathcal{F} \in \mathcal{U}$ then no algebraic curve is invariant by $\mathcal{F}$.

(iii) the set $\mathcal{U}$ is open and dense (even in the real analytic Zariski topology).

Now, for $\mathcal{F} \in \mathcal{U}$ Brunella shows that, assuming $\mathcal{F}$ admits an invariant closed positive current T, we necessarily have $Res(\mathcal{F}, T, p) = 0$ at all singularities of the foliation. This is because there are no invariant algebraic curves so that the invariant branches through a singularity do not fit into an invariant compact set (an algebraic curve). Hence the right hand side of (65) vanishes, which gives $c_1(\det N^*\mathcal{F}).[T] = 0$. But we have the relation (see [2])

$$\det N^*\mathcal{F} = K_{\mathbb{P}^n_{\mathbb{C}}} \otimes T\mathcal{F} = \mathcal{O}(-n-1) \otimes \mathcal{O}(1-d) = \mathcal{O}(-n-d). \tag{66}$$

This equality means that $c_1(\det N^*\mathcal{F}).[T] < 0$ since T is positive. This contradiction shows that that are no invariant closed positive currents, hence no invariant measures.

6. Exercises

Exercise 1.– Draw the graphs of the following functions:

$f : \mathbb{R} \to \mathbb{R}$, $f(t) = e^{-1/t}$ if $t > 0, f(t) = 0$ if $t \leq 0$. Show that $f \in C^\infty(\mathbb{R}, \mathbb{R})$.

$g : \mathbb{R} \to \mathbb{R}$, $g(t) = f(t+2)f(-t-1)$.

$h : \mathbb{R} \to \mathbb{R}$, $h(t) = \dfrac{1}{A} \displaystyle\int_{-\infty}^{t} g(s)\, ds$ where $A = \displaystyle\int_{-\infty}^{\infty} g(s)\, ds$.

Finally set $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi(x) = h(-|x|)$.

Show that all these functions are C^∞. Show that ϕ satisfies $\phi(x) = 0$ if $|x| \geq 2$, $\phi(x) = 1$ if $|x| \leq 1$ and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^n$.

Exercise 2.– (*Bump functions*). Given real numbers $0 < a < b$, $\varepsilon > 0$ and $p \in \mathbb{R}^n$ construct a function $\phi_{a,b,\varepsilon,p} : \mathbb{R}^n \to \mathbb{R}$ of class C^∞ such that, $\phi(x) = 0$ for $|x-p| \geq b$, $\phi(x) = \varepsilon$ for $|x-p| \leq a$ and $0 \leq \phi(x) \leq \varepsilon$ for all $x \in \mathbb{R}^n$.

Exercise 3.–* Find a subset of $\mathbb{R}^n$ which is NOT contained in $\mathcal{B}$!

Exercise 4.– Show the following properties of measures:

(i) If $A_1, A_2, \ldots, A_k$ is a finite collection of disjoint measurable sets then $\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$. This is called finite additivity.

(ii) If $A \subset B$, A and B measurable, then $\mu(A) \leq \mu(B)$.

(iii) If A_i is measurable and $A_1 \subset A_2 \subset A_3 \subset \cdots$, then $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$.

(iv) If A_i is measurable and $A_1 \supset A_2 \supset A_3 \supset \cdots$ and if $\mu(A_1) < \infty$, then $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$.

Exercise 5.– Show that the following affirmatives are equivalent:

(i) $L_f^{>}(t) \in \Sigma$.

(ii) $L_f^{<}(t) = \{x \in U \,:\, f(x) < t\} \in \Sigma$.

(iii) $L_f^{\geq}(t) = \{x \in U \,:\, f(x) \geq t\} \in \Sigma$.

(iv) $L_f^{\leq}(t) = \{x \in U \,:\, f(x) \leq t\} \in \Sigma$.

Hence we could have used any of these to define measurability of a function. *Hint:*

$$\{x \in U \,:\, f(x) > t\} = \bigcup_{n=1}^{\infty} \{x \in U \,:\, f(x) \geq t + 1/n\}.$$

Exercise 6.– If $\Sigma = \mathcal{B}$ in $\mathbb{R}^n$ then any continuous or lower semicontinuous or upper semicontinuous function is measurable. *Hint:* f is lower semicontinuous if $L_f^{>}(t)$ is open and upper semicontinuous if $L_f^{<}(t)$ is open.

Exercise 7.– Show that, if f and g are measurable, then so are the functions $x \mapsto af(x) + bg(x)$, $a, b \in \mathbb{C}$, $x \mapsto f(x)g(x)$, $x \mapsto |f(x)|$, $x \mapsto h(f(x))$, where $h : \mathbb{C} \longrightarrow \mathbb{C}$ is Borel measurable; $x \mapsto \text{máx}\{f(x), g(x)\}$.

Exercise 8.– Let A and B be subsets of $\mathbb{R}^n$ and define

$$A + B = \{a + b \,:\, a \in A, b \in B\}.$$

Show that if A is closed and B is compact then $A + B$ is closed. Give an example with A and B closed and $A + B$ not closed.

Exercise 9.–

(i) Show that $T(\phi) = \phi^{(k)}(0)$, $\phi \in C_c^{\infty}(\mathbb{R})$ is a distribution in $\mathcal{D}'(\mathbb{R})$ of order $\leq k$.

(ii) Show that T as in (i) is of order k in any neighborhood of 0. *Hint:* Consider $\phi_\epsilon(x) = x^k \varphi(x/\epsilon)$ where φ is a bump function equal to 1 around 0.

(iii) Show that the distribution given by $S(\phi) = \sum_0^{\infty} (-1)^k \phi^{(k)}(k)$ is not of finite order.

Exercise 10.– Show that the change from the basis $\{\partial/\partial x_j, \partial/\partial y_j\}$ to the basis $\{\partial/\partial z_j, \partial/\partial \bar{z}_j\}$ is given by the matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ -\mathrm{i}/2 & \mathrm{i}/2 \end{pmatrix} \text{ with } P^{-1} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}.$$

◁

Exercise 11.– This exercise gives an alternative description of vector bundles. Let $\{U_\alpha\}$ be an open cover of X. Suppose we are given a family of continuous functions, defined in $U_{\alpha\beta}$, $\Theta_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R})$ and satisfying $\Theta_{\alpha\beta}\Theta_{\beta\gamma}\Theta_{\gamma\alpha} = I$ in $U_{\alpha\beta\gamma}$ and $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}^{-1}$ in $U_{\alpha\beta}$ (remark that $\Theta_{\alpha\alpha} = I$). Set $\mathcal{F} = \coprod_{\alpha\in A} U_\alpha \times \mathbb{R}^n$ (disjoint union with the obviuos topology) and define the following equivalence relation in $\mathcal{F}$:

$$(\alpha, x, u) \sim (\beta, y, v) \iff x = y\,, \quad \Theta_{\alpha\beta}(x)v = u \quad \text{and } U_{\alpha\beta} \neq \emptyset.$$

Show that the quotient $\mathcal{F}/\sim$ has the structure of a real vector bundle of rank n over X, unique up to isomorphism, whose transition functions are the $\Theta_{\alpha\beta}$. ◁

Exercise 12.– If $0 \longrightarrow F \xrightarrow{f} E \xrightarrow{g} G \longrightarrow 0$ is an exact sequence of vector bundles over X, that is, $0 \longrightarrow F_x \longrightarrow E_x \longrightarrow G_x \longrightarrow 0$ is exact $\forall\, x \in X$, then f identifies F with a subbundle of E and g induces an isomorphism between E/F and G.

Exercise 13.– Determine the transition functions of $f^{-1}E$.

Exercise 14.– Show that any two hyperplanes define isomorphic bundles.

Exercise 15.– Let E be a holomorphic vector bundle of rank n over M. If E admits n linearly independent holomorphic sections, then E is holomorphically isomorphic to the trivial bundle $\underline{\mathbb{C}^n}$.

Exercise 16.– Show that the function $z \longmapsto \dfrac{1}{z}$ is locally integrable.

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