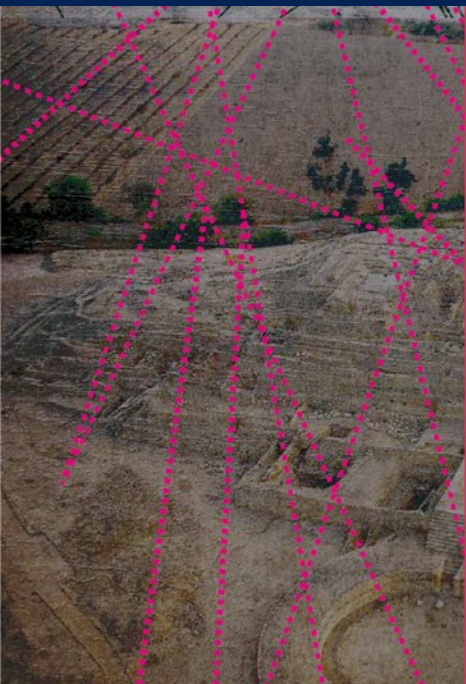
Capítulo 3



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About the Cremona group

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1 First definitions and properties

1.1 Divisors and blow-ups

Definition 1.1. — Let X be an algebraic variety. A **prime divisor** on X is an irreducible closed subset of X of codimension 1.

Examples 1.2.
- If $\dim X = 2$, *i.e.* if X is a surface, then the prime divisors of X are the irreducible curves that lie on it.
- If $X = \mathbb{P}^n_{\mathbb{C}}$, then the prime divisors are given by the zero locus of irreducible homogeneous polynomials.

Let us set

$$\mathrm{Div}(X) = \Big\{ \sum_{i=1}^{m} a_i D_i \,|\, m \in \mathbb{N},\, a_i \in \mathbb{Z},\, D_i \text{ prime divisors on } X \Big\}.$$

An element $\sum_{i=1}^{m} a_i D_i$ of $\mathrm{Div}(X)$ is **effective** if $a_i \geq 0$ for any $1 \leq i \leq m$.

If f is a non zero rational function, and D a prime divisor of X, one can define the multiplicity $\nu_f(D)$ of f at D as follows

- $\nu_f(D) = k > 0$ if f vanishes on D at the order k;
- $\nu_f(D) = -k$ if f has poles of order k on D;
- $\nu_f(D) = 0$ otherwise.

To any rational function $f \in \mathbb{C}(X)^*$ one associates a divisor $\mathrm{div}\, f \in \mathrm{Div}(X)$ defined by

$$\mathrm{div}\, f = \sum_{\substack{D \text{ prime} \\ \text{divisor}}} \nu_f(D) D.$$

Such a divisor is called a **principal divisor**. Note that a principal divisor belongs to $\mathrm{Div}(X)$ as $\nu_f(D) = 0$ for all but finitely many D. Since $\mathrm{div}\, f + \mathrm{div}\, g = \mathrm{div}\, fg$ the set of principal divisors is a subgroup of $\mathrm{Div}(X)$.

Two divisors D, D' on an algebraic variety are **linearly equivalent** if $D - D'$ is a principal divisor. The set of equivalence classes corresponds to the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors. When X is smooth, this quotient is isomorphic to the group of isomorphism classes of line bundles on X called the **Picard group** of X and denoted $\mathrm{Pic}(X)$.

Exercice 1. — Determine $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}})$.

Exercice 2. — Determine $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$.

There is a notion of intersection:

Proposition 1.3 ([26]). — *Let S be a smooth projective surface. There exists a unique bilinear symmetric form*

$$\mathrm{Div}(S) \times \mathrm{Div}(S) \to \mathbb{Z}, \qquad (D,D') \mapsto D \cdot D'$$

having the following properties:

- *if C and D are smooth curves meeting transversely, then $C \cdot D = \#(C \cap D)$;*
- *if C and C' are linearly equivalent, then $C \cdot D = C' \cdot D$.*

In particular this yields an intersection form

$$\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}, \qquad (D,D') \mapsto D \cdot D'.$$

Definition 1.4. — Let p be a point of a smooth surface S. We say that $\pi\colon Y \to S$ is a **blow-up** of $p \in S$ if

- Y is a smooth variety,
- $\pi_{|Y \smallsetminus \{\pi^{-1}(p)\}}\colon Y \smallsetminus \{\pi^{-1}(p)\} \to S \smallsetminus \{p\}$ is an isomorphism,
- $\pi^{-1}(p) \simeq \mathbb{P}^1_{\mathbb{C}}$.

The set $\pi^{-1}(p)$ is called the **exceptional divisor**.

Let us explain how to construct π. Assume for simplicity that $X = S$ is a surface. Take a neighborhood $\mathcal{U}$ of p on which there exist local coordinates x, y at p, that is the curves $x = 0$ and $y = 0$ intersects transversely at p. Up to shrinking $\mathcal{U}$ one has

$$(x = 0) \cap (y = 0) \cap \mathcal{U} = \{p\}.$$

Let us consider the subvariety $\widetilde{\mathcal{U}} \subset \mathcal{U} \times \mathbb{P}^1_{\mathbb{C}}$ defined by $xv - yu = 0$ where u and v are homogeneous coordinates on $\mathbb{P}^1_{\mathbb{C}}$. The projection $\pi\colon \widetilde{\mathcal{U}} \to \mathcal{U}$ is an isomorphism over the points of $\mathcal{U}$ where at most one of the coordinates x, y vanishes

$$\pi\big((0,y),(0:1)\big) = (0,y) \qquad \pi\big((x,0),(1:0)\big) = (x,0)$$

and $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1_{\mathbb{C}}$. It follows from the construction that the points of E can be naturally identified with the tangent directions on S at p.

Remarks 1.5. • If $\pi\colon Y\to S$ and $\pi'\colon Y'\to S$ are two blow-ups of p, then there exists an isomorphism $\varphi\colon Y\to Y'$ such that $\pi=\pi'\varphi$; we can thus speak about the blow-up of $p\in S$.

- Note that π is not an isomorphism: it contracts $E=\pi^{-1}(p)\simeq\mathbb{P}^1_{\mathbb{C}}$ onto p.

Let $\pi\colon \mathrm{Bl}_p S\to S$ be the blow-up of $p\in S$. The morphism π induces the map

$$\pi^*\colon \mathrm{Pic}(S)\to\mathrm{Pic}(\mathrm{Bl}_p S),\qquad C\mapsto\pi^{-1}C.$$

If S is a smooth algebraic surface and if $C\subset S$ is an irreducible curve, the **strict transform** of C is $\widetilde{C}=\overline{\pi^{-1}(C\smallsetminus\{p\})}$.

Let us recall that if Y is a quasi-projective variety, and if y is a point of Y, then $O_{y,Y}$ is the set of equivalence classes of pairs $(\mathcal{U},f)$ where

- $\mathcal{U}\subset Y$ is an open subset,
- $y\in\mathcal{U}$,
- $f\in\mathbb{C}[\mathcal{U}]$.

Definition 1.6. — If S is a smooth algebraic surface, $C\subset S$ a curve on S, and p a point of S, we can define the **multiplicity** $m_p(C)$ **of** C **at** p.

Let $\mathfrak{m}$ be the maximal ideal of the ring of functions $O_{p,S}$. Let f be a local equation of C; then $m_p(C)$ can be defined as the integer k such that $f\in\mathfrak{m}^k\smallsetminus\mathfrak{m}^{k+1}$. For instance if S is rational, one can find a neighborhood $\mathcal{U}$ of p in S with

$$\begin{cases}\mathcal{U}\subset\mathbb{C}^2\\ p=(0,0)\\ C \text{ is described by } \displaystyle\sum_{i=1}^m P_i(x,y)=0\end{cases}$$

where P_i denotes an homogeneous polynomial of degree i.

The multiplicity $m_p(C)$ is equal to the lowest i such that $P_i\not\equiv 0$. The following properties holds

$$\begin{cases}m_p(C)\geq 0\\ m_p(C)=0\Longleftrightarrow p\not\in C\\ m_p(C)=1\Longleftrightarrow p \text{ is a smooth point of } C\end{cases}$$

Take two distinct curves C and C' without common component. One can define an integer $(C\cdot C')_p$ which counts the intersection of C and C' at p:

- it is equal to 0 if either C, or C' does not pass through p,
- otherwise let f, resp. g be some local equation of C, resp. C' in a neighborhood of p and define $(C\cdot C')_p$ to be $\dim \frac{O_{p,S}}{(f,g)}$.

This number is related to $C\cdot C'$ as follows (*see* [26, Chapter V, Proposition 1.4]): if C and C' are two distinct curves without common irreducible component on a smooth surface, then

$$C\cdot C' = \sum_{p\in C\cap C'} (C\cdot C')_p.$$

In particular $C\cdot C'\geq 0$.

Lemma 1.7. — *Let* $\pi\colon \mathrm{Bl}_pS\to S$ *be the blow-up of* $p\in S$. *Then*

$$\pi^*C = \widetilde{C} + m_p(C)E$$

where $\widetilde{C}$ *is the strict transform of* C, *and* $E=\pi^{-1}(p)$.

Proof. Let us fix some local coordinates (x,y) such that

$$\begin{cases} p=(0,0)\\ k=m_p(C)\\ C \text{ is given by}\\ \qquad P_k(x,y)+P_{k+1}(x,y)+\ldots+P_{k+\ell}(x,y)=0\\ \text{where } P_i \text{ denotes a homogeneous polynomial of degree } i\end{cases}$$

The blow-up of p can be viewed as $(u,v)\mapsto(uv,v)$; hence the pull-back of C is given by

$$v^k\big(P_k(u,1)+vP_{k+1}(u,1)+\ldots+v^\ell P_{k+\ell}(u,1)\big)=0$$

i.e. it decomposes into k times the exceptional divisor $\pi^{-1}(0,0)=(v=0)$ and the strict transform of C. □

Let S be a compact, complex surface, and let ω_S be the line bundle of differential 2-forms on S. The **canonical divisor** K_S of S is such that $O_S(K_S)=\omega_S$.

Example 1.8. The canonical divisor of $\mathbb{P}^2_{\mathbb{C}}$ is

$$K_{\mathbb{P}^2_{\mathbb{C}}} = [-3H]$$

where H denotes a generic hyperplane of $\mathbb{P}^2_{\mathbb{C}}$.

Proposition 1.9 ([26]). — *Let S be a smooth surface, p be a point of S, and $\pi\colon \mathrm{Bl}_pS \to S$ be the blow-up of p. Set $E = \pi^{-1}(p) \simeq \mathbb{P}^1_{\mathbb{C}}$. One has*

$$\mathrm{Pic}(\mathrm{Bl}_pS) = \pi^*\mathrm{Pic}(S) + \mathbb{Z}\cdot E.$$

The intersection form on Bl_pS is induced by the intersection form on S via

$$\begin{cases} \pi^*C\cdot\pi^*C' = C\cdot C' \quad \forall C, C' \in \mathrm{Pic}(S)\\ \pi^*C\cdot E = 0 \quad \forall C \in \mathrm{Pic}(S)\\ E^2 = E\cdot E = -1\\ \widetilde{C}^2 = C^2 - 1 \quad \forall C \ni p, C \textit{ smooth} \end{cases}$$

Furthermore, $K_{\mathrm{Bl}_pS} = \pi^*K_S + E$.

The proof is decomposed in the following exercises:

Exercice 3. Prove the following equalities

$$\begin{cases} \pi^*C\cdot\pi^*C' = C\cdot C' \quad \forall C, C' \in \mathrm{Pic}(S)\\ \pi^*C\cdot E = 0 \quad \forall C \in \mathrm{Pic}(S)\\ E^2 = E\cdot E = -1\\ \widetilde{C}^2 = C^2 - 1 \quad \forall C \ni p, C \text{ smooth} \end{cases}$$

Exercice 4. Prove that

$$\mathrm{Pic}(\mathrm{Bl}_pS) = \pi^*\mathrm{Pic}(S) + \mathbb{Z}\cdot E.$$

Exercice 5. Prove that $K_{\mathrm{Bl}_pS} = \pi^*K_S + E$.

1.2 Rational and birational maps

1.2.1 First Definitions

Consider two irreducible varieties X and Y. A **rational map** $\phi\colon X \dashrightarrow Y$ is a morphism from an open subset $\mathcal{U}$ of X to Y which cannot be extended to any larger open subset; ϕ is **defined** at x if x belongs to $\mathcal{U}$. The set $X \smallsetminus \mathcal{U}$ is the **indeterminacy set** of ϕ; it is denoted $\mathrm{Ind}\,\phi$.

Suppose that $X = S$ is a smooth surface, then $\mathrm{Ind}\,\phi$ is the union of a finite number of points. One has

- if C is an irreducible curve on S, then ϕ is defined on $C \smallsetminus \mathrm{Ind}\,\phi$; the image of C is $\overline{\phi(C \smallsetminus \mathrm{Ind}\,\phi)}$ and is still denoted $\phi(C)$.

- restriction induces an isomorphism between the divisors groups of $S \smallsetminus \mathrm{Ind}\,\phi$ and S, which induces an isomorphism between $\mathrm{Pic}(S)$ and $\mathrm{Pic}(S \smallsetminus \mathrm{Ind}\,\phi)$. We can thus speak of the inverse image ϕ^*D under ϕ of a divisor D on Y.

Example 1.10. — Let $S \subset \mathbb{P}^n_{\mathbb{C}}$ be a surface, and p be a point of S. The set of lines through p can be identified with a projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$. To any point q of $S \smallsetminus \{p\}$ we associate the line through p and q; this yields a rational map $S \dashrightarrow \mathbb{P}^{n-1}_{\mathbb{C}}$ (the projection away from p). It is defined outside p and extends to a morphism $\mathrm{Bl}_p S \to \mathbb{P}^{n-1}_{\mathbb{C}}$.

A **birational map** $\phi \colon X \dashrightarrow Y$ is a rational map such that there exists a rational map $\psi \colon Y \dashrightarrow X$ such that $\phi\psi = \psi\phi = \mathrm{id}$.

1.2.2 Linear systems

Consider a divisor D on a surface S; we denote by $|D|$ the set of all effective divisors on S linearly equivalent to D. Every non-vanishing section of $\mathcal{O}_s(D)$[1] defines an element of $|D|$, namely its divisor of zeros. Conversely any element of $|D|$ is the divisor of zeros of a non-vanishing section of $\mathcal{O}_S(D)$, defined up to scalar multiplication. Therefore $|D|$ can be naturally identified with the projective space associated to the vector space[2] $\mathrm{H}^0(\mathcal{O}_S(D))$. A linear subspace $\mathscr{S}$ of $|D|$ is called a **linear system** on S; equivalently $\mathscr{S}$ can be defined by a vector subspace of $\mathrm{H}^0(\mathcal{O}_S(D))$.

The **dimension** of $\mathscr{S}$ is by definition its dimension as a projective space. A 1-dimensional linear system is a **pencil**.

A curve C is a **fixed component** of $\mathscr{S}$ if any divisor of $\mathscr{S}$ contains C.

The **fixed part** of $\mathscr{S}$ is the biggest divisor F that is contained in every element of $\mathscr{S}$.

A point p of S is a **base point** or **fixed point** of $\mathscr{S}$ if every divisor of $\mathscr{S}$ contains p. If the linear system $\mathscr{S}$ has no fixed part, then it has only a finite number, say b, of fixed points; clearly $b < D^2$, for $D \in \mathscr{S}$.

Let S be a surface. Then there is a bijection between the set

$$\big\{ \text{ rational maps } \phi \colon S \dashrightarrow \mathbb{P}^n_{\mathbb{C}} \text{ such that } \phi(S) \text{ is contained in no hyperplane } \big\}$$

and the set

$$\big\{ \text{ linear systems on } S \text{ without fixed part and of dimension } n \big\}.$$

Indeed, to the map ϕ we associate the linear system $\phi^*|H|$, where $|H|$ is the system of hyperplanes in $\mathbb{P}^n_{\mathbb{C}}$. Conversely, let $\mathscr{S}$ be a linear system on S with no fixed part and denote by $\mathscr{S}^\vee$

[1]Recall that $\mathcal{O}_S(D)$ denotes the invertible sheaf corresponding to D.

[2]Recall that $\mathrm{H}^i(\mathcal{O}_S(D))$ is the i-th cohomology group of $\mathcal{O}_S(D)$.

the projective space dual to $\mathscr{S}$. Now define a rational map $\phi\colon S \dashrightarrow \mathscr{S}^\vee$ by sending $x \in S$ to the hyperplane in $\mathscr{S}$ consisting of the divisors passing through x; the map ϕ is defined at x if and only if x is not a base point of $\mathscr{S}$.

1.2.3 Cremona maps

If $S = \mathbb{P}^2_{\mathbb{C}}$, then a birational self-map ϕ of S can be written

$$(z_0 : z_1 : z_2) \dashrightarrow \big(\phi_0(z_0,z_1,z_2) : \phi_1(z_0,z_1,z_2) : \phi_2(z_0,z_1,z_2)\big)$$

where the ϕ_i's denote homogeneous polynomials of the same degree without common factor (of positive degree). The set of all birational maps of $\mathbb{P}^2_{\mathbb{C}}$ is called the **Cremona group**, and is denoted $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. The indeterminacy set $\mathrm{Ind}\,\phi$ of ϕ is the finite set given by

$$\big\{p \in \mathbb{P}^2_{\mathbb{C}} \,|\, \phi_0(p) = \phi_1(p) = \phi_2(p) = 0\big\}.$$

The **exceptional set** $\mathrm{Exc}\,\phi$ **of** ϕ is the set of curves blown down by ϕ; one has

$$\mathrm{Exc}\,\phi = \big\{\det \mathrm{jac}\,\phi = 0\big\}.$$

The **degree of** ϕ is defined by: $\deg\phi = \deg\phi_i$. Let d be a positive integer. The set $\mathrm{Bir}_d(\mathbb{P}^2_{\mathbb{C}})$ of plane birational maps of degree d is quasi-projective: it is a Zariski open subset in the subvariety of the projective space made of triples of homogeneous polynomials of degree d modulo scalar multiplication. The group $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ acts on $\mathrm{Bir}_d(\mathbb{P}^2_{\mathbb{C}})$ as follows

$$\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) \times \mathrm{Bir}_d(\mathbb{P}^2_{\mathbb{C}}) \times \mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) \to \mathrm{Bir}_d(\mathbb{P}^2_{\mathbb{C}}), \qquad (A,\phi,B) \mapsto A\phi B^{-1}.$$

If ϕ is an element of $\mathrm{Bir}_d(\mathbb{P}^2_{\mathbb{C}})$, then $O(\phi)$ denotes the orbit of ϕ under this action.

The linear system $\mathscr{S}$ defined by any element $\phi = (\phi_0 : \phi_1 : \phi_2)$ of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is given by

$$\big\{\lambda_0\phi_0 + \lambda_1\phi_1 + \lambda_2\phi_2 = 0 \,|\, (\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2_{\mathbb{C}}\big\}.$$

It is the reciprocical image by ϕ of the net of lines

$$\big\{\lambda_0 z_0 + \lambda_1 z_1 + \lambda_2 z_2 = 0 \,|\, (\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2_{\mathbb{C}}\big\}.$$

In particular any curve of $\mathscr{S}$ is a rational one. Take a base point p of ϕ; the **multiplicity** of ϕ at p is the multiplicity of a generic curve of $\mathscr{S}$ at p, that is the order of a generic element of $\mathscr{S}$ at p.

The degree is not a birational invariant: there exist ϕ and ψ in $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ such that $\deg(\psi\phi\psi^{-1}) \neq \deg\phi$. Nevertheless the **dynamical degree**

$$\lambda(\phi) = \lim_{n\to+\infty} (\deg\phi^n)^{1/n}$$

of a birational map ϕ is. More generally consider a projective surface S, a birational self-map ϕ of S, and $||\cdot||$ any norm of the Néron-Severi real vector space $\mathrm{NS}(S)$; we can define

$$\lambda(\phi) = \lim_{n\to+\infty} ||(\phi^n)^*||^{1/n}$$

where ϕ^* is the induced action on $\mathrm{NS}(S)$.

Note that $1 \leq \lambda(\phi) \leq d$. When ϕ is an automorphism with $\lambda(\phi) > 1$, then $\lambda(\phi)$ is algebraic but never rational; in particular $\lambda(\phi) < d$. Let ω denote any Kähler form (for instance the Fubini Study form) with $\int_S \omega^2 = 1$. For any generic line L one has

$$\begin{aligned}
\lambda(\phi) &= \lim_k ||(\phi^k)^*||^{1/k} \\
&= \lim_k \Big(\int_S \beta\wedge(\phi^k)^*\beta\Big)^{1/k} \\
&= \lim_k \Big(\int_{\phi^{-k}L} \beta\Big)^{1/k} \\
&= \lim_k \big(\mathrm{vol}(\phi^{-k}L)\big)^{1/k}
\end{aligned}$$

so the dynamical degree also measures the exponential rate of growth of $(k-1)$-dimensional volume under pullback. It would be convenient if we could have $(\phi^*)^k = (\phi^k)^*$. Diller and Favre showed there is a finite sequence of blow-ups $\pi\colon S' \to S$ such that the induced map $\phi_{S'} = \pi^{-1}\phi\pi$ satisfies $(\phi^k_{S'})^* = (\phi^*_{S'})^k$ (*see* [18]). Set $\omega_{S'} = \pi^*\omega$; then

$$\begin{aligned}
\lambda(\phi) &= \lim_k \Big(\int_S \omega\wedge(\phi^k)^*\omega\Big)^{1/k} \\
&= \lim_k \Big(\int_{S'} \omega_{S'}\wedge(\phi^k_{S'})^*\omega_{S'}\Big)^{1/k} \\
&= \lim_k \Big(\int_{S'} \omega_{S'}\wedge(\phi^*_{S'})^k\omega_{S'}\Big)^{1/k}
\end{aligned}$$

The form $\omega_{S'}$ is a Kähler form so as soon as $\lambda(\phi) > 1$ the growth of $\omega_{S'}$ under $(\phi^*_{S'})^k$ gives the growth of $|(\phi^k_{S'})^*|$ and $\lambda(\phi)$ coincides with the spectral radius of $\phi^*_{S'}$, *i.e.* the modulus of the largest eigenvalue.

Definition 1.11. — Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$.

If $(\deg\phi^n)_n$ is bounded, we say that ϕ is **elliptic**.

If $(\deg\phi^n)_n$ grows linearly, then ϕ is a **Jonquières twist**.

If $(\deg\phi^n)_n$ grows quadratically, then ϕ is a **Halphen twist**.

If $(\deg\phi^n)_n$ grows exponentially, then ϕ is **hyperbolic**.

Examples 1.12.

- Birational self-maps of $\mathbb{P}^2_\mathbb{C}$ of degree 1 are maps of the type

$$\big(a_0z_0+a_1z_1+a_2z_2 : a_3z_0+a_4z_1+a_5z_2 : a_6z_0+a_7z_1+a_8z_2\big)$$

  with $\det(a_i)\neq 0$; they form the group $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$. They are elliptic maps.

- The set $\mathrm{Bir}_2(\mathbb{P}^2_\mathbb{C})$ is an irreducible algebraic variety of dimension 14. Set

$$\begin{cases} \sigma=(z_1z_2:z_0z_2:z_0z_1)\\ \rho=(z_0z_2:z_0z_1:z_2^2)\\ \tau=(z_0z_2+z_1^2:z_1z_2:z_2^2)\end{cases}$$

  One has ([11])

$$\begin{cases} \mathrm{Bir}_2(\mathbb{P}^2_\mathbb{C})=\mathcal{O}(\sigma)\cup\mathcal{O}(\phi)\cup\mathcal{O}(\tau)\\ \mathrm{Bir}_2(\mathbb{P}^2_\mathbb{C})=\overline{\mathcal{O}(\sigma)}\\ \dim\mathcal{O}(\sigma)=14,\,\dim\mathcal{O}(\rho)=13,\,\dim\mathcal{O}(\tau)=12\end{cases}$$

- Denote by $\mathcal{J}_d$ the set of birational maps of degree d of $\mathbb{P}^2_\mathbb{C}$ that preserve the pencil of lines through $p_0=(1:0:0)$. These maps are called **Jonquières maps** of degree d. The **Jonquières group** is the group $\mathcal{J}=\cup_d\mathcal{J}_d$. In affine coordinates an element ϕ of $\mathcal{J}_d$ has the following form

$$\phi(z_0,z_1)=\left(\frac{a(z_1)z_0+b(z_1)}{c(z_1)z_0+d(z_1)},\frac{\alpha z_1+\beta}{\gamma z_1+\delta}\right)$$

  with

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in\mathrm{PGL}(2,\mathbb{C})\qquad\begin{bmatrix}a(z_1) & b(z_1)\\ c(z_1) & d(z_1)\end{bmatrix}\in\mathrm{PGL}(2,\mathbb{C}(z_1)).$$

  Cleaning denominators we may assume that a, b, c and d are polynomials of respective degree $d-1$, d, $d-2$, and $d-1$. The base points of ϕ are

$$\begin{cases}\text{the point } p_0=(1:0:0) \text{ with multiplicity } d-1\\ 2d-2 \text{ single points } p_1,\, p_2,\, \ldots,\, p_{2d-2}\end{cases}$$

  The same holds for ϕ^{-1}.

  Remarks that the set of Jonquières twist is contained in $\mathcal{J}$ but the inclusion is strict (for instance σ is elliptic and belongs to $\mathcal{J}$).

- A **polynomial automorphism** ϕ of $\mathbb{C}^2$ is a bijective map of the form

$$\phi\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad (z_0,z_1) \mapsto \big(\phi_0(z_0,z_1), \phi_1(z_0,z_1)\big), \quad \phi_i \in \mathbb{C}[z_0,z_1].$$

The set of polynomial automorphisms of $\mathbb{C}^2$ form a group denoted $\mathrm{Aut}(\mathbb{C}^2)$. According to Friedland and Milnor if ϕ belongs to $\mathrm{Aut}(\mathbb{C}^2)$, then up to conjugacy ([20])

(i) either $\phi = (\alpha x + P(y), \beta y + \gamma)$ with α, β, $\gamma \in \mathbb{C}$, $\alpha\beta \neq 0$, $P \in \mathbb{C}[y]$,

(ii) or

$$\phi = h_1 h_2 \ldots h_k$$

with $h_i = (y, P_i(y) - \delta_i x)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[y]$, $\deg P_i \geq 2$.

In case (i), then ϕ is elliptic; in case (ii) ϕ is hyperbolic.

Exercice 6. — Give a description of the indeterminacy set, and the exceptional set of an automorphism of $\mathbb{P}^2_{\mathbb{C}}$.

Exercice 7. — Give a description of the indeterminacy set, and the exceptional set of σ, resp. ρ, resp. τ.

Exercice 8. — Give a description of the linear systems associated to σ, ρ and τ.

There is a "classification" of the birational maps of $\mathbb{P}^2_{\mathbb{C}}$:

Theorem 1.13 ([18, 25, 4]). — *Let ϕ be an element of the Cremona group. Then exactly one of the following holds*

- *ϕ is elliptic, furthermore either ϕ is of finite order, or ϕ is conjugate to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$;*

- *ϕ is a Jonquières twist, ϕ preserves a unique fibration that is rational and ϕ is non conjugate to an automorphism;*

- *ϕ is a Halphen twist, ϕ preserves a unique fibration that is elliptic, and ϕ is conjugate to an automorphism;*

- *ϕ is a hyperbolic map.*

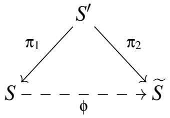
In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$.

Exercice 9. Give an example of an elliptic map, a Jonquières twist, a Halphen twist, and a hyperbolic map.

1.3 Zariski theorem

Let us recall the following statement.

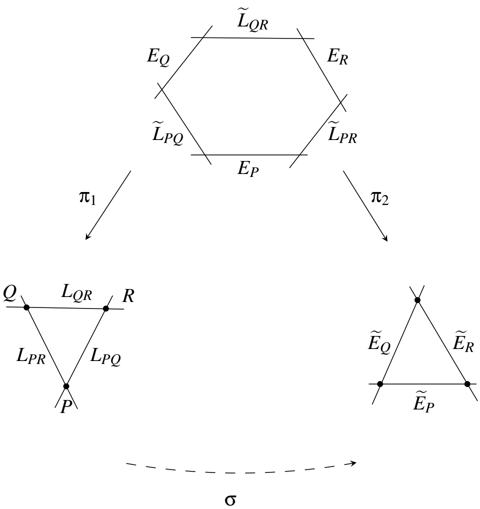
Theorem 1.14 (Zariski)**.** *Let S, $\widetilde{S}$ be two smooth projective surfaces and $\phi\colon S \dashrightarrow \widetilde{S}$ be a birational map. There exists a smooth projective surface S' and two sequences of blow-ups $\pi_1\colon S' \to S$, $\pi_2\colon S' \to \widetilde{S}$ such that $\phi = \pi_2\pi_1^{-1}$*



Example 1.15. The involution

$$\sigma\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}, \qquad\qquad (z_0 : z_1 : z_2) \dashrightarrow (z_1z_2 : z_0z_2 : z_0z_1)$$

is the composition of two sequences of blow-ups



with

$$P = (1:0:0), \qquad\qquad Q = (0:1:0), \qquad\qquad R = (0:0:1),$$

L_{PQ} (resp. L_{PR}, resp. L_{QR}) the line passing through P and Q (resp. P and R, resp. Q and R) E_P (resp. E_Q, resp. E_R) the exceptional divisor obtained by blowing up P (resp. Q, resp. R) and $\widetilde{L}_{PQ}$ (resp. $\widetilde{L}_{PR}$, resp. $\widetilde{L}_{QR}$) the strict transform of L_{PQ} (resp. L_{PR}, resp. L_{QR}).

We will prove Theorem 1.14 in the following exercises. There are two steps:

- the first one is to compose ϕ with a sequence of blow-ups in order to remove all the points of indeterminacy, we thus have

$$\begin{array}{ccc} & S' & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\widetilde{\phi}} \\ S & \overset{}{\underset{\phi}{\dashrightarrow}} & \widetilde{S} \end{array}$$

  where π_1 is a finite sequence of blow-ups and $\widetilde{\phi}$ a birational morphism;

- the second step can be stated as follows: let $\phi\colon S \to S'$ be a birational morphism between two surfaces S and S'. Assume that ϕ^{-1} is not defined at a point p of S'; then ϕ can be written $\pi\psi$ where $\pi\colon \mathrm{Bl}_p S' \to S'$ is the blow-up of $p \in S'$ and ψ a birational morphism from S to $\mathrm{Bl}_p S'$.

Remark 1.16. The first step is also possible with a rational map, and can be adapted in higher dimension whereas the second one isn't.

Exercice 10. Let $\phi\colon S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then there exists a surface S', a morphism $\eta\colon S' \to S$ which is the composition of a finite number of blow-ups, and a morphism $f\colon S' \to X$ such that

$$\begin{array}{ccc} & S' & \\ {\scriptstyle\eta}\swarrow & & \searrow{\scriptstyle f} \\ S & \underset{\phi}{\dashrightarrow} & X \end{array}$$

commutes.

The second step is decomposed in the two following exercises.

Exercice 11. Let $\phi\colon S \dashrightarrow S'$ be a birational map between two surfaces S and S'. If there exists a point $p \in S$ such that ϕ is not defined at p there exists a curve $\mathcal{C}$ on S' such that $\phi^{-1}(\mathcal{C}) = p$.

Exercice 12. Let $\phi\colon S \to S'$ be a birational morphism between two surfaces S and S'. Assume that ϕ^{-1} is not defined at a point p of S'; then ϕ can be written $\pi\psi$ where $\pi\colon \mathrm{Bl}_pS' \to S'$ is the blow-up of $p \in S'$ and ψ a birational morphism from S to Bl_pS'

$$\begin{array}{ccc} & \mathrm{Bl}_pS' & \\ {}^{\psi}\nearrow & & \searrow^{\pi} \\ S & \xrightarrow[\phi]{} & S' \end{array}$$

1.4 Exceptional configurations and characteristic matrices

Let $\phi \in \mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ be a birational map of degree ν. By Theorem 1.14 there exist a smooth projective surface S' and π, η two sequences of blow-ups such that

$$\begin{array}{ccc} & S & \\ {}^{\pi}\swarrow & & \searrow^{\eta} \\ \mathbb{P}^2_\mathbb{C} & \overset{}{\underset{\phi}{-\,-\,-\,-\,\to}} & \mathbb{P}^2_\mathbb{C} \end{array}$$

We can rewrite π as follows

$$\pi\colon S = S_k \stackrel{\pi_k}{\to} S_{k-1} \stackrel{\pi_{k-1}}{\to} \ldots \stackrel{\pi_2}{\to} S_1 \stackrel{\pi_1}{\to} S_0 = \mathbb{P}^2_\mathbb{C}$$

where π_i is the blow-up of the point p_{i-1} in S_{i-1}. Let us set

$$\mathrm{E}_i = \pi_i^{-1}(p_i), \qquad\qquad \mathcal{E}_i = (\pi_{i+1} \circ \ldots \circ \pi_k)^*\mathrm{E}_i.$$

The divisors $\mathcal{E}_i$ are called the **exceptional configurations** of π and the p_i base-points of ϕ.

An **ordered resolution** of ϕ is a decomposition $\phi = \eta\pi^{-1}$ where η and π are ordered sequences of blow-ups. An ordered resolution of ϕ induces two basis of $\mathrm{Pic}(S)$

- $\mathcal{B} = \big\{e_0 = \pi^*\mathrm{H}, e_1 = [\mathcal{E}_1], \ldots, e_k = [\mathcal{E}_k]\big\}$,
- $\mathcal{B}' = \big\{e'_0 = \eta^*\mathrm{H}, e'_1 = [\mathcal{E}'_1], \ldots, e'_k = [\mathcal{E}'_k]\big\}$,

where H is a generic line. We can write e'_i as follows

$$e'_0 = \nu e_0 - \sum_{i=1}^k m_i e_i, \qquad\qquad e'_j = \nu_j e_0 - \sum_{i=1}^k m_{ij} e_i,\ j \geq 1.$$

The matrix of change of basis

$$M = \begin{bmatrix} \nu & \nu_1 & \dots & \nu_k \\ -m_1 & -m_{11} & \dots & -m_{1k} \\ \vdots & \vdots & & \vdots \\ -m_k & -m_{k1} & \dots & -m_{kk} \end{bmatrix}$$

is called **characteristic matrix** of ϕ. The first column of M, which is the **characteristic vector** of ϕ, is the vector $(\nu, -m_1, \dots, -m_k)$. The other columns

$$(\nu_i, -m_{1i}, \dots, -m_{ki})$$

describe the "behavior of $\mathcal{E}_i'$": if $\nu_j > 0$, then $\pi(\mathcal{E}_j')$ is a curve of degree ν_j in $\mathbb{P}^2_{\mathbb{C}}$ through the points p_ℓ of ϕ with multiplicity $m_{\ell j}$.

Example 1.17. Consider the birational map

$$\sigma\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}, \qquad (z_0 : z_1 : z_2) \dashrightarrow (z_1 z_2 : z_0 z_2 : z_0 z_1).$$

The points of indeterminacy of σ are

$$P = (1:0:0), \quad Q = (0:1:0), \quad R = (0:0:1)$$

and the exceptional set is the union of the three following lines

$$\Delta = \{z_0 = 0\}, \quad \Delta' = \{z_1 = 0\}, \quad \Delta'' = \{z_2 = 0\}.$$

First we blow up P; let us denote E the exceptional divisor and $\mathcal{D}_1$ the strict transform of $\mathcal{D}$. Set

$$\begin{cases} z_1 = u_1 \\ z_2 = u_1 v_1 \end{cases} \qquad \begin{cases} z_1 = r_1 s_1 \\ z_2 = s_1 \end{cases}$$

In the coordinates (u_1, v_1) (resp. (r_1, s_1)) the exceptional divisor E is given by $\{u_1 = 0\}$ (resp. $\{s_1 = 0\}$) and Δ_1'' (resp. Δ_1') by $\{v_1 = 0\}$ (resp. $\{r_1 = 0\}$).

On the one hand

$$(u_1, v_1) \to (u_1, u_1 v_1)_{(z_1,z_2)} \to (u_1 v_1 : v_1 : 1) = \left(\frac{1}{u_1}, \frac{1}{u_1 v_1}\right)_{(z_1,z_2)} \to \left(\frac{1}{u_1}, \frac{1}{v_1}\right)_{(u_1,v_1)}$$

and on the other hand

$$(r_1,s_1)\to(r_1s_1,s_1)_{(z_1,z_2)}\to(r_1s_1:1:r_1)=\left(\frac{1}{r_1s_1},\frac{1}{s_1}\right)_{(z_1,z_2)}\to\left(\frac{1}{r_1},\frac{1}{s_1}\right)_{(r_1,s_1)}.$$

Hence E is sent on Δ_1; as σ is an involution Δ_1 is sent on E.

Now blow up Q_1; this time let us denote F the exceptional divisor and $\mathcal{D}_2$ the strict transform of $\mathcal{D}_1$:

$$\begin{cases} z_0 = u_2 \\ z_2 = u_2v_2 \end{cases} \qquad\qquad \begin{cases} z_0 = r_2s_2 \\ z_2 = s_2 \end{cases}$$

In the coordinates (u_2,v_2) (resp. (r_2,s_2)) one has $\mathrm{F}=\{u_2=0\}$ and $\Delta_2''=\{v_2=0\}$ (resp. $\mathrm{F}=\{s_2=0\}$ and $\Delta_2=\{r_2=0\}$).

We have

$$(u_2,v_2)\to(u_2,u_2v_2)_{(z_0,z_2)}\to(v_2:u_2v_2:1)=\left(\frac{1}{u_2},\frac{1}{u_2v_2}\right)_{(z_0,z_2)}\to\left(\frac{1}{u_2},\frac{1}{v_2}\right)_{(u_2,v_2)}$$

and

$$(r_2,s_2)\to(r_2s_2,s_2)_{(z_0,z_2)}\to(1:r_2s_2:r_2)=\left(\frac{1}{r_2s_2},\frac{1}{s_2}\right)_{(z_0,z_2)}\to\left(\frac{1}{r_2},\frac{1}{s_2}\right)_{(r_2,s_2)}.$$

Therefore F is sent on Δ_2' and Δ_2' on F.

Finally we blow up R_2; let us denote G the exceptional divisor and set

$$\begin{cases} z_0 = u_3 \\ z_1 = u_3v_3 \end{cases} \qquad\qquad \begin{cases} z_0 = r_3s_3 \\ z_2 = s_3 \end{cases}$$

Note that

$$(u_3,v_3)\to(u_3,u_3v_3)_{(z_0,z_1)}\to(v_3:1:u_3v_3)=\left(\frac{1}{u_3},\frac{1}{u_3v_3}\right)_{(z_0,z_1)}\to\left(\frac{1}{u_3},\frac{1}{v_3}\right)_{(u_3,v_3)}$$

and

$$(r_3,s_3)\to(r_3s_3,s_3)_{(z_0,z_1)}\to(1:r_3:r_3s_3)=\left(\frac{1}{r_3s_3},\frac{1}{s_3}\right)_{(z_0,z_1)}\to\left(\frac{1}{r_3},\frac{1}{s_3}\right)_{(r_3,s_3)}.$$

One has $\mathrm{G}=\{u_3=0\}$ and $\Delta_3'=\{v_3=0\}$ (resp. $\mathrm{G}=\{s_3=0\}$ and $\Delta_3=\{r_3=0\}$).

Thus $\mathrm{G}\to\Delta_3'$ and $\Delta_3'\to\mathrm{G}$. There are no more point of indeterminacy, no more exceptional curve; in other words σ is conjugate to an automorphism of $\mathrm{Bl}_{P,Q_1,R_2}\mathbb{P}^2_{\mathbb{C}}$.

Let H be a generic line. Note that $\mathcal{E}_1 = \mathrm{E}$, $\mathcal{E}_2 = \mathrm{F}$, $\mathcal{E}_3 = \mathrm{H}$. Consider the basis $\{\mathrm{H}, \mathrm{E}, \mathrm{F}, \mathrm{G}\}$. After the first blow-up Δ and E are swapped; the point blown up is the intersection of Δ' and Δ'' so $\Delta \to \Delta + \mathrm{F} + \mathrm{G}$. Then $\sigma^*\mathrm{E} = \mathrm{H} - \mathrm{F} - \mathrm{G}$. Similarly we have

$$\begin{cases} \sigma^*\mathrm{F} = \mathrm{H} - \mathrm{E} - \mathrm{G} \\ \sigma^*\mathrm{G} = \mathrm{H} - \mathrm{E} - \mathrm{F} \end{cases}$$

It remains to determine $\sigma^*\mathrm{H}$. The image of a generic line by σ is a conic hence $\sigma^*\mathrm{H} = 2\mathrm{H} - m_1\mathrm{E} - m_2\mathrm{F} - m_3\mathrm{G}$. Let L be a generic line given by $a_0z_0 + a_1z_1 + a_2z_2$. A computation shows that

$$(u_1, v_1) \to (u_1, u_1v_1)_{(z_1,z_2)} \to (u_1^2v_1 : u_1v_1 : u_1) \to u_1(a_0v_2 + a_1u_2v_2 + a_2)$$

vanishes to order 1 on $\mathrm{E} = \{u_1 = 0\}$ thus $m_1 = 1$. Note also that

$$(u_2, v_2) \to (u_2, u_2v_2)_{(z_0,z_2)} \to (u_2v_2 : u_2^2v_2 : u_2) \to u_2(a_0v_2 + a_1u_2v_2 + a_2),$$

respectively

$$(u_3, v_3) \to (u_3, u_3v_3)_{(z_0,z_1)} \to (u_3v_3 : u_3 : u_3^2v_3) \to u_3(a_0v_3 + a_1 + a_2u_3v_3)$$

vanishes to order 1 on $\mathrm{F} = \{u_2 = 0\}$, resp. $\mathrm{G} = \{u_3 = 0\}$ so $m_2 = 1$, resp. $m_3 = 1$. Therefore $\sigma^*\mathrm{H} = 2\mathrm{H} - \mathrm{E} - \mathrm{F} - \mathrm{G}$ and the characteristic matrix of σ in the basis $\{\mathrm{H}, \mathrm{E}, \mathrm{F}, \mathrm{G}\}$ is

$$M_\sigma = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Exercice 13. Let us consider the involution given by

$$\rho\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}, \qquad (z_0 : z_1 : z_2) \dashrightarrow (z_0z_1 : z_2^2 : z_1z_2).$$

We can show that $M_\rho = M_\sigma$.

Exercice 14. Consider the birational map

$$\tau\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}, \qquad (z_0 : z_1 : z_2) \dashrightarrow (z_0^2 : z_0z_1 : z_1^2 - z_0z_2).$$

We can verify that $M_\tau = M_\sigma$.

Solution 1. — Let us determine $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}})$. Consider the homomorphism of groups given by

$$\theta\colon \mathrm{Div}(\mathbb{P}^n_{\mathbb{C}}) \to \mathbb{Z}, \qquad D \mapsto \deg D.$$

Let D be in $\ker\theta$; write D as $\sum_i a_iD_i$ where D_i denotes a prime divisor given by a homogeneous polynomial $f_i \in \mathbb{C}[z_0,z_1,\ldots,z_n]$ of some degree d_i. Since $\sum_i a_id_i = 0$ one has: $f = \prod_i f_i^{a_i}$ belongs to $\mathbb{C}(\mathbb{P}^n_{\mathbb{C}})^*$, and by construction $D = \mathrm{div}\, f$ so D is a prime divisor.

Conversely any prime divisor is equal to $\mathrm{div}\,\frac{g}{h}$ where g, h are polynomials of the same degree; any principal divisor thus belongs to $\ker\theta$.

In other words $\ker\theta$ is the subgroup of principal divisors. So $\mathrm{Div}(\mathbb{P}^n_{\mathbb{C}})\big/_{\ker\theta} \simeq \mathbb{Z}$.

Solution 2. — Let us determine $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}})$? Set

$$h_1 = \{0\}\times\mathbb{P}^1_{\mathbb{C}} \qquad h_2 = \mathbb{P}^1_{\mathbb{C}}\times\{0\} \qquad \mathcal{U} = \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}} \smallsetminus (h_1\cup h_2).$$

Since $\mathcal{U}$ is isomorphic to the affine space $\mathbb{A}^2$, every divisor on $\mathcal{U}$ is the divisor of a rational function. Let us consider a divisor on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, then $D_{|\mathcal{U}} = \mathrm{div}\,\phi$ so

$$D = \mathrm{div}\,\phi + nh_1 + mh_2$$

for some integers n and m. Furthermore $D \sim nh_1+mh_2$. Hence $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}})$ is generated by the classes of h_1 and h_2. Obviously $h_1\cdot h_2 = 1$. Moreover

$$h_1\cdot h_1 \sim h_1\cdot(\{\infty\}\times\mathbb{P}^1_{\mathbb{C}})$$

as $h_1 \sim \{\infty\}\times\mathbb{P}^1_{\mathbb{C}}$. Since $h_1\cap(\{\infty\}\times\mathbb{P}^1_{\mathbb{C}}) = \emptyset$ one gets $h_1^2 = 0$. Similarly $h_2^2 = 0$.

Solution 3. We can replace C and C' by linearly equivalent divisors and so assume that p lies on no component of C nor C'. Therefore obviously $\pi^*C\cdot\pi^*C' = C\cdot C'$, and $\pi^*C\cdot E = 0$.

Take C a curve passing through p with multiplicity 1. Its strict transform $\widetilde{C}$ meets E transversely at one point which corresponds in E to the tangent direction defined at p by C. Thus $C\cdot E = 1$. From $\widetilde{C} = \pi^*C - E$ (Lemma 1.7) and $\pi^*C\cdot E = 0$ we get $E^2 = -1$.

Solution 4. Let us prove that

$$\phi\colon \mathrm{Pic}(S)\oplus\mathbb{Z} \to \mathrm{Pic}(\mathrm{Bl}_pS) \qquad (D,n)\mapsto \pi^*D + nE$$

is an isomorphism. Every irreducible curve on Bl_pS except E is a strict transform of its image in S, hence ϕ is surjective. Assume that there is a divisor D on S such that $\pi^*D + nE = 0$. Taking the intersection with E we get that $n = 0$ and upon applying π_* we see that $D = 0$.

Solution 5. Recall that if $D = \sum_i a_i D_i$ is a divisor, and if all the a_i are non zero, the **support** $\mathrm{Supp}\,D$ of D is $\cup_i D_i$.

Consider a differential form $\omega \in \Omega^2(S)$ such that p does not belong to $\mathrm{Supp}(\mathrm{div}\,\omega)$. Since $\pi\colon \mathrm{Bl}_pS \smallsetminus E \to S \smallsetminus \{p\}$ is an isomorphism, obviously $\mathrm{div}(\pi^*\omega) = \pi^*(\mathrm{div}\omega)$ over $\mathrm{Bl}_pS \smallsetminus E$. If x and y are local parameters at p then $\omega = f\,dx \wedge dy$ where f denotes an element of $\mathcal{O}_p$ such that $f(p) \neq 0$. Let us blow up p: set

$$\begin{cases} x = u \\ y = uv \end{cases}$$

Then $\pi^*\omega = \pi^*(f)u\,du \wedge dv$ on S, and since $\pi^*(f) \neq 0$ on E we get

$$\mathrm{div}(\pi^*\omega) = \pi^*(\mathrm{div}\omega) + E$$

that is $K_{\mathrm{Bl}_pS} = \pi^*K_S + E$.

Solution 6. — Any element ϕ of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ satisfies $\mathrm{Ind}\,\phi = \mathrm{Exc}\,\phi = \emptyset$.

Solution 7. — One has

$$\begin{cases} \mathrm{Ind}\,\sigma = \big\{(1:0:0), (0:1:0), (0:0:1)\big\}, \mathrm{Exc}\,\sigma = \big\{z_0 = 0\big\} \cup \big\{z_1 = 0\big\} \cup \big\{z_2 = 0\big\} \\ \\ \mathrm{Ind}\,\rho = \big\{(1:0:0), (0:1:0)\big\}, \mathrm{Exc}\,\rho = \big\{z_0 = 0\big\} \cup \big\{z_2 = 0\big\} \\ \\ \mathrm{Ind}\,\tau = \big\{(1:0:0)\big\}, \mathrm{Exc}\,\tau = \big\{z_2 = 0\big\} \end{cases}$$

Solution 8. — The linear system defined by σ is the set of conics in $\mathbb{P}^2_{\mathbb{C}}$ passing through $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.

The linear system defined by ρ is the set of conics in $\mathbb{P}^2_{\mathbb{C}}$ passing through $(1:0:0)$, $(0:1:0)$ and tangent to $z_2 = 0$.

The linear system defined by τ is the set of conics in $\mathbb{P}^2_{\mathbb{C}}$ passing through $(1:0:0)$ that are tangent to $z_2 = 0$, and osculate it.

Solution 9. —

- Any birational map of finite order is elliptic; any element of the following groups

  $$\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}), \quad \big\{(\alpha z_0 + P(z_1), \beta z_1 + \gamma) \,|\, \alpha, \beta \in \mathbb{C}^*, \gamma \in \mathbb{C}, P \in \mathbb{C}[z_1]\big\}$$

  is elliptic.

- Any element of $\mathcal{J}$ of the form
$$\left(\frac{a(z_1)z_0+b(z_1)}{c(z_1)z_0+d(z_1)},z_1\right)$$
with $\frac{(\mathrm{tr}M)^2}{\det M}\in\mathbb{C}(z_1)\smallsetminus\mathbb{C}$ where M denotes the matrix defined by
$$\left[\begin{array}{cc} a(z_1) & b(z_1)\\ c(z_1) & d(z_1)\end{array}\right]$$
is a Jonquières twist ([10]).

- Let ϕ be the birational self-map of $\mathbb{P}^2_{\mathbb{C}}$ given by
$$\phi=(z_0z_2^2+z_1^3-2z_1z_2^2:z_1z_2^2:z_0z_2^2+z_1^3+z_1z_2^2-z_2^3).$$
One has $\deg\phi^n\sim n^2$.

- Consider the family of birational maps (f_ε) given by ([18])
$$f_\varepsilon=\left(z_1+1-\varepsilon,z_0\frac{z_1-\varepsilon}{z_1+1}\right).$$
If

  - $\varepsilon=-1$, then f_ε is elliptic,
  - $\varepsilon\in\{0,1\}$, then f_ε is a Jonquières twist,
  - $\varepsilon\in\{1/2,1/3\}$, then f_ε is a Halphen twist,
  - $\varepsilon\in\{\cup_{k\geq 4}1/k\}$, then f_ε is hyperbolic.

Solution 10 ([2], Theorem 2.7)**.** As X lies in some projective space, one can assume that $X=\mathbb{P}^m_{\mathbb{C}}$. Of course one can suppose that $\phi(S)$ lies in no hyperplane of $\mathbb{P}^m_{\mathbb{C}}$. Hence ϕ corresponds to a linear system $\mathscr{S}\subset|D|$ of dimension m on S.

If $\mathscr{S}$ has no base point, then ϕ is a morphism, and we are done.

Consider now the case where ϕ has a base point p_1. Let $\pi_1\colon \mathrm{Bl}_{p_1}S\to S$ be the blow-up of p_1. Then the exceptional curve E_1 occurs in the fixed part of the linear system $\pi_1^*\mathscr{S}\subset|\pi_1^*D|$ with some multiplicity $k_1\geq 1$. That is the system $\mathscr{S}_1\subset|\pi_1^*D-k_1E_1|$ obtained by subtracting k_1E_1 from each element of $\pi_1^*\mathscr{S}$ has no fixed component. It thus defines a rational map $\phi_1=\phi\pi_1\colon S_1\dashrightarrow\mathbb{P}^m_{\mathbb{C}}$. If ϕ_1 is a morphism, then we are done. Otherwise we repeat the process. Hence by induction we get a sequence $\pi_n \circ \pi_{n-1} \circ \ldots \circ \pi_1$ of blow-ups and a linear system $\mathscr{S}_n \subset |D_n = \pi_n^* D_{n-1} - k_n E_n|$ on S_n with no fixed part. Note that

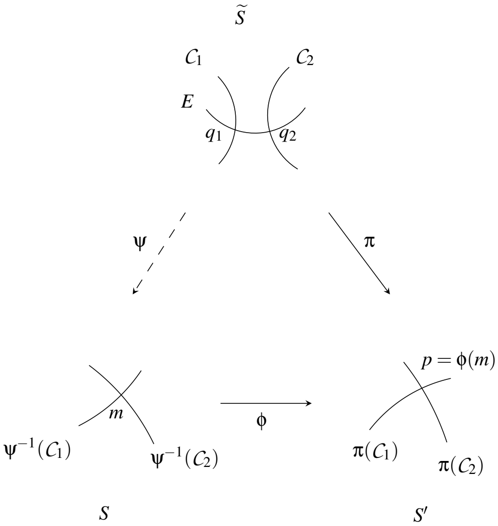
$$D_n^2 = D_{n-1}^2 - k_n^2 < D_{n-1}^2.$$

Since $\mathscr{S}_k$ has no fixed part $D_k^2 \geq 0$ for all k and so a finite number of blow-ups is needed. In other words after a finite number of blow-ups one gets a linear system with no base points which defines a morphism $S_N \to \mathbb{P}^m_{\mathbb{C}}$.

Solution 11 ([2], Lemma 2.9)**.** Suppose S affine, with $\pi^{-1}(p) \neq \emptyset$, so that there is an embedding $\iota\colon S \hookrightarrow \mathbb{A}^n$. The rational map $\iota \circ \phi^{-1}\colon S' \dashrightarrow \mathbb{A}^n$ is defined by rational functions ψ_1, $\ldots$, ψ_n; furthermore one of them, for instance ψ_1, is not defined at p, that is $\psi_1 \notin O_{S',p}$. One can write ψ_1 as $\frac{u}{v}$ with u, v in $O_{S',p}$, u and v coprime, and $v(p) = 0$. Let us consider the curve $\mathcal{C}$ on S defined by $\phi^* v = 0$. Denote by x_1 the first coordinate function on $S \subset \mathbb{A}^n$; on S one has $\phi^* u = x_1 \phi^* v$. It follows that $\phi^* u = \phi^* v = 0$ on $\mathcal{C}$ so that $\mathcal{C} = \phi^{-1}(\{u = v = 0\})$. Since u and v are coprime the set $\{u = v = 0\}$ is finite. Shrinking S' if needed one can assume that $\{u = v = 0\} = \{p\}$, and thus $\mathcal{C} = \phi^{-1}(p)$.

Solution 12 ([30])**.** Assume that $\psi = \pi^{-1}\phi$ is not a morphism. Let m be a point of S such that ψ is not defined at m. On the one hand $\phi(m) = p$ and ϕ is not locally invertible at m, on the other hand there exists a curve in $\mathrm{Bl}_p S'$ contracted on m by ψ^{-1} (Exercise 11). This curve is necessarily the exceptional divisor E obtained by blowing up.

Let q_1, q_2 be two different points of E at which ψ^{-1} is well defined and let $\mathcal{C}_1$, $\mathcal{C}_2$ be two germs of smooth curves transverse to E. Then $\pi(\mathcal{C}_1)$ and $\pi(\mathcal{C}_2)$ are two germs of smooth curve transverse at p which are the image by ϕ of two germs of curves at m. The differential of ϕ at m is thus of rank 2: contradiction with the fact that ϕ is not locally invertible at m.



Solution 13. — We can show that $M_\rho = M_\sigma$.

Solution 14. — We can verify that $M_\tau = M_\sigma$.

2 Generation of the Cremona group in any dimension

2.1 In dimension 2

Recall that σ and ρ are the elements of $\mathrm{Bir}(\mathrm{P}^2_{\mathbb{C}})$ given by

$$\sigma = (z_1z_2 : z_0z_2 : z_0z_1), \qquad \rho = (z_0z_2 : z_0z_1 : z_2^2).$$

Theorem 2.1 ([31, 9]). — *The group* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *is generated by* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}(3,\mathbb{C})$ *and* σ*:*

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}}) = \langle \mathrm{PGL}(3,\mathbb{C}), \sigma\rangle.$$

Let us remark that $\sigma = (z_1 : z_2 : z_0)\rho(z_1 : z_2 : z_0)\rho$, hence

$$\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}}) = \langle \mathrm{PGL}(3,\mathbb{C}), \rho\rangle.$$

Definition 2.2. — Let ϕ_0, ϕ_1, ..., $\phi_n \in \mathbb{C}(z_0,z_1,\ldots,z_n)$ be some rational functions; we define

$$\mathrm{jac}(\phi_0,\phi_1,\ldots,\phi_n) = \det\left(\left[\frac{\partial\phi_i}{\partial z_j}\right]_{0\leq i,\,j\leq n}\right) \in \mathbb{C}(z_0,z_1,\ldots,z_n).$$

Definition 2.3. — If $\phi = (\phi_0 : \phi_1 : \ldots : \phi_n)$ is a birational self-map of $\mathbb{P}^n_{\mathbb{C}}$, the **jacobian determinant** of ϕ is defined to be $\mathrm{jac}(\phi_0,\phi_1,\ldots,\phi_n)$. It is defined up to multiplication with the $(n+1)$-th power of an element of $\mathbb{C}^*$, and has degree $(n+1)(d-1)$.

Remark 2.4. — The jacobian determinant of $\phi \in \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ is a polynomial which determines the hypersurfaces of $\mathbb{P}^n_{\mathbb{C}}$ where the map ϕ is not locally an isomorphism.

One can check that $\det\mathrm{jac}\,\tau$ is a perfect cube, and the jacobian determinant of any element ϕ in $\langle \mathrm{PGL}(3,\mathbb{C}), \tau\rangle$ is a perfect cube ([24]); therefore

$$\langle \mathrm{PGL}(3,\mathbb{C}), \tau\rangle \subsetneq \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}}).$$

Alexander showed Theorem 2.1; we will follow its proof ([1]). Let us first introduce some definitions and notations. Let us consider a birational map ϕ of $\mathbb{P}^2_{\mathbb{C}}$ of degree $d > 1$ (note that if $d = 1$, then according to Lemma 2.5 the map ϕ is an automorphism of $\mathbb{P}^2_{\mathbb{C}}$, and thus satisfies Theorem 2.1). Denote by p_0, p_1, ..., p_k the base points of ϕ, and by m_i the multiplicity of p_i. Assume up to reindexation that

$$m_0 \geq m_1 \geq \ldots \geq m_k.$$

Let S be a surface, and let p be a point of S. The exceptional divisor obtained by blowing up p is called **first infinitesimal neighborhood**, and the points of E are called **infinitely near** p.

The k**-th infinitesimal neighborhood of** p is the set of points contained in the first infinitesimal neighborhood of a point of the $(k-1)$-th infinitesimal neighborhood of p. On the contrary the points of S are called **proper point**. The **general quadratic birational map** centered at p, q, and r is the application (defined up to automorphism) $\psi \in O(\sigma)$ such that $\mathrm{Ind}\,\psi = \{p, q, r\}$.

In his proof Noether showed that for any $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ one can find a general quadratic birational map ψ such that $\deg\phi\psi < \deg\phi$, and so by induction proved that $\phi = \psi_1\psi_2\ldots\psi_\ell$ up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ where ψ_i are general quadratic birational maps. But it is false for instance if one of the base points is proper and the others in its infinitesimal neighborhoods. To give a complete proof Alexander introduces the **complexity of the linear system** associated to ϕ defined by

$$2c = d - m_0.$$

Geometrically it is the number of points except p_0 that belong to the intersection of a generic line through p_0 and a curve of the linear system. Denote by C the set of points defined by

$$C = \{p_i \,|\, i \geq 1, m_i > c\}$$

and by n the cardinal of C. Alexander's idea is the following: apply to ϕ a sequence of general quadratic birational maps in order to decrease the complexity c until $c = 1$ and the cardinal n until $n = 0$.

Lemma 2.5. — *Let ϕ be a birational self-map of $\mathbb{P}^2_{\mathbb{C}}$ of degree d. Let p_0, p_1, …, p_k be the base points of ϕ, and m_0, m_1, …, m_k be their multiplicity. Then*

$$\sum_{j=0}^{k} m_j^2 = d^2 - 1 \tag{2.1}$$

$$\sum_{j=0}^{k} m_j(m_j-1) = (d-1)(d-2) \tag{2.2}$$

$$\sum_{j=0}^{k} m_j = 3d - 3 \tag{2.3}$$

Proof. One gets (2.3) from (2.1) and (2.2) as follows :

$$\begin{aligned}\sum_{j=0}^{k} m_j &= -\sum_{j=0}^{k} m_j(m_j-1) + \sum_{j=0}^{k} m_j^2 \\ &= -(d-1)(d-2) + d^2 - 1 \\ &= 3d - 3\end{aligned}$$

Exercice 15. — Prove relation (2.1).

Exercice 16. — Prove equality (2.2).

□

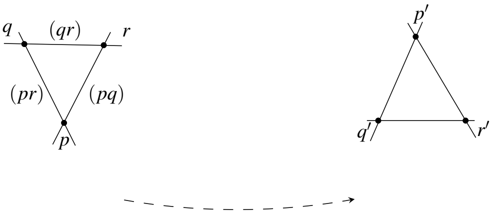
Exercice 17. Prove that $2c \geq 0$.

Exercice 18. Prove the following inequality: $2c \geq 1$.

Exercice 19. Prove that

$$2c \geq m_1 \geq m_2 \geq \ldots \geq m_n > c.$$

Take a general quadratic birational map ψ centered at p, q, and r; the lines (pq), (qr), and (pr) are blown down by ψ onto r', p', and q':



Lemma 2.6. *If $d > 1$, then $n \geq 2$. Hence $m_0 > \frac{d}{3}$.*

Furthermore if $n \geq 3$, then the points p_i with $i \in \{1, 2, \ldots, k\}$ are not all aligned.

Exercice 20. Prove Lemma 2.6

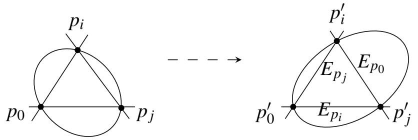
Lemma 2.7. *Compose ϕ with a general quadratic birational map centered at p_0, q, and r; the complexity of the system is constant if and only if the point p_0' is the point of maximal multiplicity. Otherwise the complexity of the system decreases.*

Exercice 21. Prove Lemma 2.7

Lemma 2.8. *Assume that there exist two points p_i and p_j in C which are not infinitely near, and not infinitely near p_0. After composition with a general quadratic map centered at p_0, p_i, and p_j then*

- *either the complexity of the system decreases,*
- *or the cardinal of C decreases by 2.*

Proof. Let us compose ϕ with a general quadratic birational map whose base points are p_0, p_i and p_j



Denote by L' the new linear system; the degree d' of L' is

$$d' = 2d - m_0 - m_i - m_j$$

furthermore

$$\begin{cases} m'_j = d - m_0 - m_i < c \\ m'_0 = d - m_i - m_j \\ m'_i = d - m_0 - m_j < c \end{cases}$$

Let C' be the set of base points with multiplicity strictly larger than c'. One has

$$d' = d + (d - m_0 - m_i - m_j) = d + (2c - m_i - m_j).$$

In particular $d' < d$.

After this composition

- p_0, p_i, and p_j are not base points anymore (they have been blown up on lines);
- the other base points don't change, and their multiplicity remains constant;
- there are three new base points p'_0, p'_i, and p'_j.

The multiplicity of the new base points is equal to the number of intersections (counted with multiplicity) of the corresponding line (that is contracted) and the strict transform of a general curve of the linear system. According to Bezout theorem one has

$$\begin{cases} m'_0 = d - m_i - m_j \\ m'_i = d - m_0 - m_j \\ m'_j = d - m_0 - m_i \end{cases}$$

Let us now distinguish two cases: the case where p'_0 is not the point of highest multiplicity and the case where it is:

- if p'_0 is not the point of highest multiplicity, then the complexity of the system decreases (Lemma 2.7);
- otherwise p'_0 is the point of highest multiplicity, then the complexity of the system remains constant (Lemma 2.7). According to Lemma 2.6 the point p'_0 belongs to C'. Moreover since $m_i > c$, $m_j > c$, and $d - m_0 = 2c$ then $m'_i < c$, $m'_j < c$ that is p'_i and p'_j don't belong to C'. Hence $n' = n - 2$.

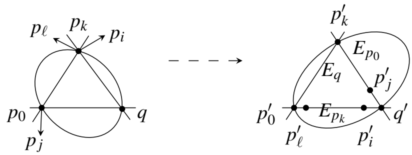
□

Lemma 2.9. *Suppose that there exists a base point p_k in C which is not infinitely near p_0. After composition with a general quadratic birational map*

- *there is no infinitely near base points above p_0 (resp. p_k),*
- *there is no infinitely near base points above p'_0,*
- *the complexity of the linear system remains constant,*
- *the cardinal of C remains constant.*

Proof. Let us compose ϕ with a general quadratic birational map centered at p_0, p_k, and q such that

- the lines (p_0q) and (p_kq) don't contain base points;
- there is no base point infinitely near p_k in the direction of the line (p_kq);
- there is no base point infinitely near p_0 in the direction of the line (p_0q).



Remark that the degree increases; indeed, the degree of the new system is

$$d' = 2d - m_0 - m_k = d + 2c - m_k \geq d$$

and

$$\begin{cases} m'_0 = d - m_k \geq d - m_0 = 2c \geq m_1 \\ m'_q = d - m_0 - m_k = 2c - m_k < c \\ m'_k = d - m_0 = 2c > c \end{cases}$$

In particular the base point p_0' is the point of highest multiplicity. The complexity remains constant (Lemma 2.7). The cardinal of C is equal to the cardinal of C': we blow up two points of C and get two new points.

We don't transform a point infinitely near p_k (resp. p_0) in a point infinitely near p_0' nor q'. Indeed assume by contradiction that we transform a point p_i infinitely near p_k in a point infinitely near q'. It means that p_k is in the direction of the line (p_0p_k). Denoting by D the divisor representing (p_0p_k) one has

$$(C\cdot D)_{p_k} = m_k + m_i$$

so

$$C\cdot D = m_0 + m_k + m_i > m_0 + 2c = d$$

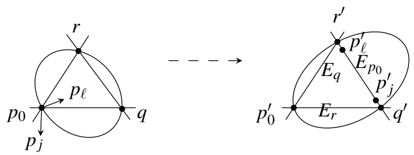
which is impossible by Bezout theorem. The same holds if we consider a point infinitely near p_0. □

Lemma 2.10. *Assume that all points of C are infinitely near p_0. After composing with a general quadratic birational map*

- *there is no infinitely near base point above the point of highest multiplicity p_0',*
- *the complexity of the linear system remains constant,*
- *the cardinal of C decreases by 2.*

Proof. Compose ϕ with a general quadratic birational map centered at p_0, r, and q such that

- the lines (p_0r), (p_0q), and (rq) don't contain base points of the new system;
- the lines (p_0r), (p_0q), and (rq) are not in the direction of the points infinitely near p_0.



The degree strictly increases; indeed $d' = 2d - m_0 > d$. Since the elements of the linear system don't pass through r and q hence according to Bezout theorem p_0' is a point of multiplicity d. It is thus the point of highest multiplicity. Moreover the complexity of the system is

$$2c' = 2d - m_0 - d = d - m_0 = 2c.$$

Any curve of the linear system intersects (p_0r) and (p_0q) at $d-m_0=2c$ points so q' and r' become base points of the system, and $m'_r=m'_q=2c>c=c'$. As a consequence $n'=n+2$.

The points infinitely near p_0 are dispersed on the line $(r'q')$; thanks to the assumption on the line (rq) there is no base point infinitely near p'_0. □

Proof of Theorem 2.1. Let us first describe the two keysteps:

Step a: if there is one base point in C that is not infinitely near the base point p_0 of highest multiplicity go to "Step b"; otherwise let us apply Lemma 2.10 to ϕ. We thus get that there is no more infinitely near base points above p'_0, and n increases by 2. Then since there is no more infinitely near base points above p'_0 one can apply Lemma 2.9 until all the points of C are distinct. The complexity and the number of base points with multiplicity $>c$ except p'_0 remain constant (still by Lemma 2.9). But now $n\geq 3$ and so the base points of C are not aligned (Lemma 2.6). Take two points p_i and p_j such that p_ℓ and p_q don't belong to (p'_0p_i), (p'_0p_j) and (p_ip_j). Let us now apply two times Lemma 2.8 to p_ℓ and p_q. If the complexity decreases come back to the beginning of "Step a"; otherwise $n+2$ decreases by 4 and p'_0 has no more infinitely near base points with multiplicity $>c$ so let us go on with "Step b".

Step b is decomposed in two cases:

- either C contains two base points that aren't infinitely near and one applies Lemma 2.8; if the complexity decreases come to "Step a", otherwise come back to the beginning of "Step b";
- or one applies Lemma 2.9 then the base points are "separated" and one comes back to "Step b".

Using this strategy one gets first that the complexity decreases until 1, and then that the cardinal of C is zero. We thus have a system with at most one base point p'_0, *i.e.* using Lemma 2.5 and the definition of c the two following equalities hold

$$\begin{cases} m_0 = 3d-3 \\ 1 = d-m_0 \end{cases}$$

Therefore $d=1$ and $m_0=0$, that is after composing ϕ with well choosen general quadratic birational maps ϕ is an automorphism of $\mathbb{P}^2_{\mathbb{C}}$. □

2.2 In higher dimensions

Theorem 2.11 ([27, 32]). — *Let $n \geq 3$ be an integer. Any set of generators of* $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ *contains an infinite uncountable number of elements of* $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}}) \smallsetminus \mathrm{Aut}(\mathbb{P}^n_{\mathbb{C}})$.

We follow Cantat's notes based on the proof of Pan ([32]).

2.2.1 Exceptional hypersurfaces

Definition 2.12. — Let ϕ be a birational map of $\mathbb{P}^n_{\mathbb{C}}$, and let X be an irreducible hypersurface of $\mathbb{P}^n_{\mathbb{C}}$. We say that X is **ϕ-exceptional** if there exists an open subset $\mathcal{U}$ of X which is mapped onto a subset of codimension ≥ 2 by ϕ.

Lemma 2.13. — *Let ϕ_1, ϕ_2, …, ϕ_m be some birational self-maps of $\mathbb{P}^n_{\mathbb{C}}$. Consider*

$$\phi = \phi_m \phi_{m-1} \dots \phi_1.$$

The irreducible hypersurface X of $\mathbb{P}^n_{\mathbb{C}}$ is ϕ-exceptional if there exist an integer i between 1 *and m, and a ϕ_i-exceptional hypersurface X_i such that X_i is birational equivalent to X.*

2.2.2 Jonquières maps with prescribed exceptional set

Consider the homogeneous coordinates $(z_0 : z_1 : \ldots : z_{n-1})$ on $\mathbb{P}^{n-1}_{\mathbb{C}}$, and the homogeneous coordinates $(u : v)$ on $\mathbb{P}^1_{\mathbb{C}}$. Let Y be an irreducible hypersurface of degree d in $\mathbb{P}^{n-1}_{\mathbb{C}}$, distinct from $z_0 = 0$. Assume that $h = 0$ is a reduced homogeneous equation of Y. Consider the birational map

$$\psi_Y : \mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \dashrightarrow \mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$$

defined by

$$\big((z_0 : z_1 : \ldots : z_{n-1}), (u : v)\big) \dashrightarrow \big((z_0 : z_1 : \ldots : z_{n-1}), (uz_0^d : vh(z_0, z_1, \ldots, z_{n-1})\big).$$

The map ψ_Y is birational, and ψ_Y contracts the generic points of $Y \times \mathbb{P}^1_{\mathbb{C}}$ onto the codimension 2 subset $Y \times \{(1:0)\}$ of $\mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

The projective variety $\mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is birationally equivalent to $\mathbb{P}^n_{\mathbb{C}}$; an explicit birational map from $\mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{P}^n_{\mathbb{C}}$ is

$$\eta \colon \mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}, \quad \big((z_0 : z_1 : \ldots : z_{n-1}), (u : v)\big) \dashrightarrow \big(uz_0 : vz_0 : vz_1 : \ldots : vz_{n-1}\big).$$

Conjugate ψ_Y by η, and set $X = \eta(Y \times \mathbb{P}^1_{\mathbb{C}})$; since η blows down

$$\big(Y \times \{(1:0)\}\big) \smallsetminus \{u = 0\}$$

onto $(1:0:0:\ldots:0) \in \mathbb{P}^n_{\mathbb{C}}$ one gets:

Lemma 2.14. — *For any irreducible hypersurface Y of $\mathbb{P}^{n-1}_{\mathbb{C}}$ of degree d there exist a birational self-map ϕ_Y of $\mathbb{P}^n_{\mathbb{C}}$ of degree $d+1$, and a hypersurface X of $\mathbb{P}^n_{\mathbb{C}}$ such that*

- *X is birationally equivalent to $Y \times \mathbb{P}^1_{\mathbb{C}}$,*
- *X is ϕ_Y-exceptional.*

In case $n = 3$ the previous statement says that: for any irreducible curve C in $\mathbb{P}^2_{\mathbb{C}}$ of degree ℓ there exists $\phi_C \in \mathrm{Bir}(\mathbb{P}^3_{\mathbb{C}})$ of degree $d+1$, and a hypersurface X in $\mathbb{P}^3_{\mathbb{C}}$ such that

- X is birationally equivalent to $C \times \mathbb{P}^1_{\mathbb{C}}$,
- and X is ϕ_C-exceptional.

Consider now the particular case of smooth plane cubics: the set of these curves is a one-parameter family so according to Lemma 2.13 one gets Theorem 2.11 for $n = 3$. More generally one concludes as follows.

2.2.3 Stable equivalence

Definition 2.15. — Let Y, and Y' be two varieties; Y is m-**stably equivalent** to Y' if there exists a birational map from $Y \times \mathbb{P}^m_{\mathbb{C}}$ to $Y' \times \mathbb{P}^m_{\mathbb{C}}$.

Remark 2.16. — Be careful there exist complex projective varieties Y of dimension $n \geq 3$ such that Y is not rational but Y is stably equivalent to $\mathbb{P}^n_{\mathbb{C}}$.

Lemma 2.17. — *Let Y and Y' be two smooth irreducible hypersurfaces of $\mathbb{P}^{n-1}_{\mathbb{C}}$ of degree $\geq n+1$. If Y and Y' are m-stably equivalent, then Y and Y' are isomorphic.*

Lemmas 2.13, 2.14, and 2.17 imply Theorem 2.11.

2.2.4 A similar argument to Gizatullin's one

Let us consider the birational involution σ_n of $\mathbb{P}^n_{\mathbb{C}}$ defined by

$$\sigma_n = \Big(\prod_{\substack{i=0\\ i\neq 0}}^{n} z_i : \prod_{\substack{i=0\\ i\neq 1}}^{n} z_i : \ldots : \prod_{\substack{i=0\\ i\neq n}}^{n} z_i\Big).$$

Definition 2.18. — A **monomial map** of $\mathbb{P}^n_{\mathbb{C}}$ is a birational self-map of $\mathbb{P}^n_{\mathbb{C}}$ of the form

$$\big(\alpha_1 z_1^{a_{11}} z_2^{a_{12}} \ldots z_n^{a_{1n}}, \alpha_2 z_1^{a_{21}} z_2^{a_{22}} \ldots z_n^{a_{2n}}, \ldots, \alpha_n z_1^{a_{n1}} z_2^{a_{n2}} \ldots z_n^{a_{nn}}\big)$$

in the affine chart $z_0 = 1$ with $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{C}^*)^n$ and $\big[a_{ij}\big]_{1 \leq i,j \leq n} \in \mathrm{GL}(n, \mathbb{Z})$.

Blanc and Heden prove that $\langle\sigma_n,\,\mathrm{Aut}(\mathbb{P}^n_\mathbb{C})\rangle\neq\mathrm{Bir}(\mathbb{P}^n_\mathbb{C})$ for n odd:

Theorem 2.19 ([5]). — *If n is odd, there are monomial maps of $\mathbb{P}^n_\mathbb{C}$ which do not belong to $\langle\sigma_n,\,\mathrm{Aut}(\mathbb{P}^n_\mathbb{C})\rangle$.*

The idea of the proof is the same as Gizatullin's. They prove the following statement:

Proposition 2.20 ([5]). — *Assume n odd. The jacobian determinant of any element of $\langle\sigma_n,\,\mathrm{Aut}(\mathbb{P}^n_\mathbb{C})\rangle$ is equal to αP^2 for some $\alpha\in\mathbb{C}^*$ and some homogeneous polynomial $P\in\mathbb{C}[z_0,z_1,\ldots,z_n]$.*

Corollary 2.21. — *Suppose n odd. The quadratic birational involution of $\mathbb{P}^n_\mathbb{C}$ given by*

$$\big(z_1z_2:z_0z_1:z_0z_2:\ldots:z_0z_n\big)$$

does not belong to $\langle\sigma_n,\,\mathrm{Aut}(\mathbb{P}^n_\mathbb{C})\rangle$.

Exercice 22. Let $\psi\in\mathbb{C}[z_0,z_1,\ldots,z_n]_d$ be a homogeneous polynomial of degree $d\in\mathbb{N}$, and ϕ_0, ϕ_1, …, $\phi_n\in\mathbb{C}(z_0,z_1,\ldots,z_n)_e$ be homogeneous rational functions of degree $e\in\mathbb{Z}\smallsetminus\{0\}$. Prove that

$$\mathrm{jac}(\psi\phi_0,\psi\phi_1,\ldots,\psi\phi_n)=\left(1+\frac{d}{e}\right)\mathrm{jac}(\phi_0,\phi_1,\ldots,\phi_n)\psi^{n+1}.$$

Exercice 23. Using Exercice 22 prove that $\mathrm{jac}\,\sigma_n=n(-1)^n\displaystyle\prod_{i=0}^{n}z_i^{n-1}$.

Exercice 24. Let $\phi=(\phi_0:\phi_1:\ldots:\phi_n)$ and $\psi=(\psi_0:\psi_1:\ldots:\psi_n)$ be two birational self-maps of $\mathbb{P}^n_\mathbb{C}$. Set $d_1=\deg\phi$, and $d_2=\deg\psi$.

Assume that $\deg(\phi\psi)=d_1d_2$; then the chain rule states that

$$\mathrm{jac}(\phi\psi)=\psi^*(\mathrm{jac}\,\phi)\,\mathrm{jac}\,\psi$$

where $\psi^*(\mathrm{jac}\,\phi)$ is obtained by replacing each z_i with ψ_i in $\mathrm{jac}\,\phi$.

If $\deg(\phi\psi)=d_1d_2-m$ for $m>0$ there exists a homogeneous polynomial Q of degree m that divides the formal composition of ϕ and ψ. Prove that

$$\mathrm{jac}(\phi\psi)=\left(\frac{d_1d_2-m}{d_1d_2}\right)\frac{\psi^*(\mathrm{jac}\,\phi)\,\mathrm{jac}\,\psi}{Q^{n+1}}.$$

Deduce from it the Proposition 2.20.

Exercice 25. Prove Corollary 2.21 : compute the jacobian determinant of

$$\big(z_1z_2:z_0z_1:z_0z_2:\ldots:z_0z_n\big)$$

and conclude with Proposition 2.20.

Solution 15. — Let $\mathscr{S}$ be the linear system defined by ϕ. Consider two curves C and D of $\mathscr{S}$. According to Bezout theorem one has $C\cdot D = d^2$. Blow up $\mathbb{P}^2_{\mathbb{C}}$ at p_0, and denote by C', resp. D' the strict transform of C, resp. D; according to Lemma 1.7

$$C'\cdot D' = (\pi^*C - m_0E)\cdot(\pi^*D - m_0E)$$

so

$$C'\cdot D' = \pi^*C\cdot\pi^*D - \pi^*C\cdot m_0E - m_0E\cdot\pi^*D + m_0E\cdot m_0E$$

that is

$$C'\cdot D' = \pi^*C\cdot\pi^*D - \pi^*C\cdot m_0E - m_0E\cdot\pi^*D - m_0^2$$

hence

$$C'\cdot D' = C\cdot D - m_0^2$$

and finally

$$d^2 = C\cdot D = C'\cdot D' + m_0^2.$$

The points p_1, p_2, $\ldots$, p_k are still points of multiplicity m_1, m_2, $\ldots$, m_k. By induction one has

$$d^2 = \widetilde{C}\cdot\widetilde{D} + \sum_{j=0}^{k} m_j^2$$

where $\widetilde{C}$, resp. $\widetilde{D}$ is the strict transform of C, resp. D after the blow up of p_0, p_1, $\ldots$, p_k. Moreover the curves $\widetilde{C}$ and $\widetilde{D}$ intersect at only one point that does not belong to $\{p_0, p_1, \ldots, p_k\}$; hence $\widetilde{C}\cdot\widetilde{D} = 1$. Therefore

$$d^2 = 1 + \sum_{j=0}^{k} m_j^2.$$

Solution 16. — Consider a curve C in $\mathbb{P}^2_{\mathbb{C}}$ that belongs to the linear system defined by ϕ. Let $\pi\colon \mathrm{Bl}_{p_0}\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be the blow-up of p_0, and C' be the strict transform of C. One has (Proposition 1.9)

$$K_{\mathrm{Bl}_{p_0}\mathbb{P}^2_{\mathbb{C}}} = \pi^* K_{\mathbb{P}^2_{\mathbb{C}}} + E$$

and so

$$K_{\mathrm{Bl}_{p_0}\mathbb{P}^2_{\mathbb{C}}}\cdot C' = \pi^* K_{\mathbb{P}^2_{\mathbb{C}}}\cdot C + m_0.$$

By induction one gets

$$K_S\cdot\widetilde{C} = K_{\mathbb{P}^2_{\mathbb{C}}}\cdot C + \sum_{j=0}^{k} m_j$$

where $S=\mathrm{Bl}_{p_0,p_1,\ldots,p_k}\mathbb{P}^2_\mathbb{C}$, and $\widetilde{C}$ is the strict transform of C. The curve $\widetilde{C}$ is smooth so according to Riemann-Roch theorem and adjunction formula one obtains

$$K_S\cdot\widetilde{C}=2g(\widetilde{C})-2-\widetilde{C}^2$$

where $g(C)$ denotes the real genus of C, that is the topological genus of a desingularization of C. So

$$K_{\mathbb{P}^2_\mathbb{C}}\cdot C+\sum_{j=0}^k m_j=2g(\widetilde{C})-2-\widetilde{C}^2.$$

Since $g(\widetilde{C})=0$ one has

$$K_{\mathbb{P}^2_\mathbb{C}}\cdot C+\sum_{j=0}^k m_j=-2-\widetilde{C}^2 \tag{2.4}$$

But $\widetilde{C}^2=C^2-\sum_{j=0}^k m_j^2$ and $K_{\mathbb{P}^2_\mathbb{C}}=-3H$ thus

$$\begin{aligned}(2.4)\quad &\Leftrightarrow\quad -3d+\sum_{j=0}^k m_j=-2-C^2+\sum_{j=0}^k m_j^2\\ &\Leftrightarrow\quad -3d+\sum_{j=0}^k m_j=-2-d^2+\sum_{j=0}^k m_j^2\\ &\Leftrightarrow\quad d^2-3d+2=\sum_{j=0}^k m_j(m_j-1)\\ &\Leftrightarrow\quad (d-1)(d-2)=\sum_{j=0}^k m_j(m_j-1)\end{aligned}$$

Solution 17. The degree of elements of the linear system defined by ϕ is d, hence the multiplicity of a point is bounded by d.

Solution 18. If an homogeneous polynomial P of degree d has a point of multiplicity d then $(P=0)$ is not irreducible, it is the union of d lines.

Solution 19. According to Bezout's theorem the line through p_0 and p_1 intersects any curve of the linear system at d points counted with multiplicity. But the line through p_0 and p_1 intersects any curve of the system at p_0 with multiplicity m_0 so $m_1\leq d-m_0=2c$. We thus have the inequalities

$$2c\geq m_1\geq m_2\geq\ldots\geq m_n>c.$$

Solution 20. One has: $c\,(2.2)-(c-1)\,(2.1)$ gives on the one hand

$$c\sum_{i=0}^{k} m_i(m_i-1)-(c-1)\sum_{i=0}^{k} m_i^2=\sum_{i=0}^{k} m_i(m_i-c)$$

and on the other hand

$$c(d-1)(d-2)-(c-1)(d^2-1)=(d-1)(d-3c+1).$$

Hence

$$\sum_{i=0}^{k} m_i(m_i-c)=(d-1)(d-3c+1) \tag{2.5}$$

Since $3c-1\geq \frac{1}{2}>0$ and $m_{n+i}-c<0$ for all $i>0$ then

$$\sum_{i=0}^{n} m_i(m_i-c)\geq \sum_{i=0}^{k} m_i(m_i-c)$$

and according to 2.5

$$\sum_{i=0}^{n} m_i(m_i-c)\geq (d-1)(d-3c+1)$$

so $\sum_{i=0}^{n} m_i(m_i-c)>d(d-2c)=d(m_0-c)$. But

$$\sum_{i=0}^{n} m_i(m_i-c)=m_0(m_0-c)+\sum_{i=1}^{n} m_i(m_i-c)$$

therefore

$$\sum_{i=1}^{n} m_i(m_i-c)>d(m_0-c)-m_0(m_0-c)=(d-m_0)(m_0-c)=2c(m_0-c).$$

Since $2c\geq m_1\geq m_2\geq \ldots\geq m_n\geq c$ one has

$$2c\sum_{i=1}^{n}(m_i-c)>2c(m_0-c)$$

and as $c>0$

$$\sum_{i=1}^{n}(m_i-c)>m_0-c.$$

But $m_1\leq m_0$ thus $n\geq 2$.

From $m_0\geq m_i$ for all i one has

$$0\geq \sum_{i=0}^{n} m_i(m_i-m_0)=\sum_{i=0}^{n} m_i^2-m_0\sum_{i=0}^{n} m_i=(d-1)(d+1-3m_0).$$

So $d+1-3m_0\leq 0$, and $m_0>\frac{d}{3}$.

Solution 21. The complexity of the system after composing with a general quadratic birational map centered at p_0, q, and r is

$$\begin{aligned} 2c' = d' - m'_{\max} &= 2d - m_0 - m_q - m_r - m'_{\max} \\ &= d - m_0 + m'_0 - m'_{\max} \\ &= 2c + m'_0 - m'_{\max} \end{aligned}$$

where $m'_{\max}$ denotes the highest multiplicity of the base points of the new system. Therefore $c' \leq c$ and $c = c'$ if and only if $m'_0 = m'_{\max}$.

Solution 22. *See* [5, Lemma 2.3]

Solution 23. [5, Corollary 2.4] Since $\sigma_n = \left(\frac{\psi}{z_0} : \frac{\psi}{z_1} : \ldots : \frac{\psi}{z_n}\right)$ with $\psi = \prod_{i=0}^{n-1}(z_i)^{n-1}$. It follows by Exercice 22 that

$$\mathrm{jac}(\sigma_n) = \left(1 + \frac{n+1}{-1}\right) \mathrm{jac}\big(z_0^{-1}, z_1^{-1}, \ldots, z_n^{-1}\big)\, \psi^{n+1} = n(-1)^n \prod_{i=0}^{n}(z_i)^{n-1}.$$

Solution 24. [5, Proposition 2.6] The formula

$$\mathrm{jac}(\phi\psi) = \left(\frac{d_1 d_2 - m}{d_1 d_2}\right) \frac{\psi^*(\mathrm{jac}\,\phi)\,\mathrm{jac}\,\psi}{Q^{n+1}}$$

directly follows from Exercice 22.

Since n is odd, we see that if the result is true for ϕ and ψ, then it is true for the composition $\phi\psi$. It remains to note that

- as we have seen $\mathrm{jac}(\sigma_n) = n(-1)^n \prod_{i=0}^{n}(z_i)^{n-1}$, that is $\mathrm{jac}(\sigma_n)$ is a square multiplied by a constant when n is odd,
- if ϕ is an automorphism of $\mathbb{P}^n_{\mathbb{C}}$, thenjac$(\phi)$ belongs to $\mathbb{C}$.

Solution 25. [5, Corollary 2.7] Since

$$\mathrm{jac}\big(z_1 z_2 : z_0 z_1 : z_0 z_2 : \ldots : z_0 z_n\big) = -2 z_0^{n-1} z_1 z_2$$

the result follows then from Proposition 2.20.

3 Action of the Cremona group on the Picard-Manin space and applications

3.1 Picard-Manin space and Bubble space

Let S, S_i be some complex projective surfaces. Any $\pi_i\colon S_i \to S$ birational morphism induces an embedding

$$\pi^*\colon \mathrm{NS}(S) \to \mathrm{NS}(S_i)$$

of Néron-Severi groups. We say that π_2 is **above** π_1 if $\pi_1^{-1}\pi_2$ is regular. Starting with two birational morphisms one can always find a third one that covers the two first. Therefore the inductive limit of all groups $\mathrm{NS}(S_i)$ for all surfaces S_i above S is well-defined. It is the **Picard-Manin space** $\mathcal{Z}_S$ of S. Structures invariant by the morphisms π_i^* go through the limit and so $\mathcal{Z}_S$ is provided with

- an intersection form,
- a nef cone $\mathcal{Z}_S^+ = \lim\limits_{\rightarrow} \mathrm{NS}^+(S_i)$,
- a canonical class which can be seen as a linear form $\mathcal{Z}_S \to \mathbb{Z}$.

Consider all surfaces S_i above S that is all birational morphisms $\pi_i\colon S_i \to S$. Take $\pi_1\colon S_1 \to S$, $\pi_2\colon S_2 \to S$, and $p_1 \in S_1$, $p_2 \in S_2$. The point p_1 is **identified with** p_2 if $\pi_1^{-1}\pi_2$ is a local isomorphism that sends p_2 onto p_1. The **Bubble space** $\mathcal{B}(S)$ of S is the union of all points of all surfaces above S modulo the equivalence relation induced by this identification.

If $p \in \mathcal{B}(S)$ is represented by a point p on a surface $S_i \to S$ we denote by e_p the divisor class of the exceptional divisor of the blow-up of p. Then

$$\begin{cases} e_p \cdot e_p = -1 \\ e_p \cdot e_q = 0 \text{ if } p \neq q \end{cases}$$

Exercice 26. — Prove the previous formulas in case where p is a point of $\mathbb{P}^2_{\mathbb{C}}$, $S_1 = \mathrm{Bl}_p\mathbb{P}^2_{\mathbb{C}}$, q is a point on E_p, and $S_2 = \mathrm{Bl}_q S_1$.

Embed $\mathrm{NS}(S)$ as a subgroup of $\mathcal{Z}_S$. This finite dimensional lattice is orthogonal to e_p for any $p \in \mathcal{B}(S)$. Furthermore

$$\mathcal{Z}_S = \big\{D + \sum_{p \in \mathcal{B}(S)} a_p e_p \,|\, D \in \mathrm{NS}(S),\, a_p \in \mathbb{R}\big\}$$

note that $a_p = 0$ except finitely many. The **completed Picard-Manin space** $\overline{\mathcal{Z}_S}$ of S is the L^2-completion of $\mathcal{Z}_S$, that is

$$\overline{\mathcal{Z}}_S = \big\{D + \sum_{p\in\mathcal{B}(S)} a_p e_p \,|\, D \in \mathrm{NS}(S),\, a_p \in \mathbb{R},\, \sum a_p^2 < \infty\big\}.$$

Furthermore the intersection form on $\mathrm{NS}(S_i)$ induces an intersection form with signature $(1,\infty)$ on $\overline{\mathcal{Z}}_S$. Let $\overline{\mathcal{Z}}_S^+$ be the **nef cone** of $\overline{\mathcal{Z}}_S$, and $\mathcal{L}\overline{\mathcal{Z}}_S = \big\{d \in \overline{\mathcal{Z}}_S \,|\, d\cdot d = 0\big\}$ be the **light cone** of $\mathcal{Z}_S$.

3.2 Hyperbolic space and isometries

The **hyperbolic space** $\mathbb{H}_S$ of S is then defined by

$$\mathbb{H}_S = \big\{d \in \overline{\mathcal{Z}}_S^+ \,|\, d\cdot d = 1\big\}.$$

Note that $\mathbb{H}_S$ is an infinite dimensional analogue of the classical hyperbolic space $\mathbb{H}^n$. The distance on $\mathbb{H}_S$ is defined by: for $d,\, d' \in \mathbb{H}_S$

$$\cosh(\mathrm{dist}(d,d')) = d\cdot d'.$$

The **geodesics** are intersections of $\mathbb{H}_S$ with planes. The projection $\mathbb{H}_S \to \mathbb{P}(\overline{\mathcal{Z}}_S)$ is one-to-one, the boundary of its image is the projection of the cone of isotropic vectors of $\overline{\mathcal{Z}}_S$. Hence

$$\partial\mathbb{H}_S = \big\{\mathbb{R}^+ d \,|\, d \in \overline{\mathcal{Z}}_S^+,\, d\cdot d = 0\big\}.$$

If $\pi\colon S' \to S$ is a birational morphism, we get an isometry π^* (and not simply an embedding) between $\mathbb{H}_S$ and $\mathbb{H}_{S'}$. This allows to define an action of $\mathrm{Bir}(S)$ on $\mathbb{H}_S$. Let $\phi\colon S \to S$ be a birational map; there exists S' a surface and $\pi_1\colon S' \to S$, $\pi_2\colon S' \to S$ two birational morphisms such that $\phi = \pi_2\pi_1^{-1}$ (*see for example* [2]). One can define the isometry $\phi_\bullet$ of $\mathbb{H}_S$ by

$$\phi_\bullet = (\pi_2^*)^{-1}\pi_1^*.$$

The isometries of $\mathbb{H}_S$ are classified in three types ([6, 23]). The **translation length** of an isometry $\phi_\bullet$ of $\mathbb{H}_S$ is defined by

$$L(\phi_\bullet) = \inf\big\{\mathrm{dist}(p,\phi_\bullet(p)) \,|\, p \in \mathbb{H}_S\big\}.$$

If the infimum is a minimum, then

- either it is equal to 0 and $\phi_\bullet$ has a fixed point in $\mathbb{H}_S$, $\phi_\bullet$ is thus **elliptic**,

- or it is positive and $\phi_\bullet$ is **hyperbolic**. Hence the set of points $p \in \mathbb{H}_S$ such that $\mathrm{dist}(p, \phi_\bullet(p))$ is equal to $L(\phi_\bullet)$ is a geodesic line $\mathrm{Ax}(\phi_\bullet) \subset \mathbb{H}_S$. Its boundary points are represented by isotropic vectors $\omega(\phi_\bullet)$ and $\alpha(\phi_\bullet)$ in $\overline{\mathcal{Z}}_S$ such that

$$\phi_\bullet(\omega(\phi_\bullet)) = \lambda(\phi)\,\omega(\phi_\bullet) \qquad\qquad \phi_\bullet(\alpha(\phi_\bullet)) = \frac{1}{\lambda(\phi)}\alpha(\phi_\bullet).$$

The axis of $\phi_\bullet$ is the intersection of $\mathbb{H}_S$ with the plane containing $\omega(\phi_\bullet)$ and $\alpha(\phi_\bullet)$. Forall $p \in \mathbb{H}_S$ one has

$$\lim_{k\to+\infty} \frac{\phi_\bullet^{-k}(p)}{\lambda(\phi)} = \alpha(\phi_\bullet) \qquad\qquad \lim_{k\to+\infty} \frac{\phi_\bullet^{k}(p)}{\lambda(\phi)} = \omega(\phi_\bullet).$$

When the infimum is not realized, $L(\phi_\bullet) = 0$ and $\phi_\bullet$ is **parabolic**: $\phi_\bullet$ fixes a unique line in $\mathcal{L}\overline{\mathcal{Z}}_S$; this line is fixed pointwise, and all orbits $\phi_\bullet^n(p)$ in $\mathbb{H}_S$ accumulate to the corresponding boundary point when n goes to $\pm\infty$.

Exercice 27. — Let $\phi_\bullet$ be a hyperbolic isometry; it acts as a translation along $\mathrm{Ax}(\phi_\bullet)$. Let us prove that this length of translation is $L(\phi_\bullet) = \log\lambda(\phi)$.

One can normalize $\alpha(\phi_\bullet)$ and $\omega(\phi_\bullet)$ such that $\alpha(\phi_\bullet) = \omega(\phi_\bullet) = \frac{1}{2}$; one has

$$\mathrm{Ax}(\phi_\bullet) = \big\{u\alpha(\phi_\bullet) + v\omega(\phi_\bullet) \,|\, uv = 1\big\}.$$

Set $p = \alpha(\phi_\bullet) + \omega(\phi_\bullet)$; the point p lies on $\mathrm{Ax}(\phi_\bullet)$. Compute $2\cosh(\mathrm{dist}(p, \phi_\bullet(p)))$, and $2\cosh(L(\phi_\bullet))$. Conclude.

There is a strong relationship between classification of birational maps of $\mathbb{P}^2_\mathbb{C}$ and the classification of isometries of $\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$:

Theorem 3.1 ([7]). — *Let ϕ be a birational map of the complex projective plane. Then*

- *ϕ is a elliptic map if and only if $\phi_\bullet$ is an elliptic isometry;*
- *ϕ is a twist if and only if $\phi_\bullet$ is a parabolic isometry;*
- *ϕ is a hyperbolic map if and only if $\phi_\bullet$ is a hyperbolic isometry.*

Remark 3.2. — Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$, and let h be the class of a line viewed as a point in $\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$. Then

$$\phi_\bullet(h) = (\deg\phi)h - \sum a_p e_p$$

where a_p is the multiplicity of the linear system $\phi_*|O(1)|$ at the point p. Since h does not intersect any of the e_p one gets

$$\cosh(\mathrm{dist}(h,\phi_\bullet(h))) = h\cdot\phi_\bullet(h) = \deg\phi$$

this establishes a link between $\deg\phi^n$ and $\mathrm{dist}\big(h,\phi_\bullet^n(h)\big)$.

Exercice 28. — Take a generic element ϕ in $\mathrm{Bir}_2(\mathbb{P}^2_\mathbb{C})$. Then

$$\begin{cases} \mathrm{Ind}\,\phi = \{p_0,\, p_1,\, p_2\}, & \mathrm{Exc}\,\phi = \{L_{p_0p_1}, L_{p_1p_2}, L_{p_0p_2}\}, \\ \mathrm{Ind}\,\phi^{-1} = \{q_0,\, q_1,\, q_2\}, & \mathrm{Exc}\,\phi^{-1} = \{L_{q_0q_1}, L_{q_1q_2}, L_{q_0q_2}\} \end{cases}$$

Let h be the class of a line in $\mathbb{P}^2_\mathbb{C}$. Determine $\phi_\bullet(h)$.

Assume ϕ is an isomorphism on a neighborhood of p, and $\phi(p) = q$; determine $\phi_\bullet(e_p)$.

Suppose $L_{q_1q_2}$ is blown down onto p_0 by ϕ^{-1} ; determine $\phi_\bullet(e_{p_0})$.

Exercice 29. — Any set $\{p_0 = (1:0:0),\, p_1,\, p_2\}$ of three distinct and non colinear points is the indeterminacy set of a Jonquières map of degree 2. Any set $\{p_0 = (1:0:0),\, p_1,\, p_2,\, p_3\}$ of four distinct points such that

- no three of them are on a line through p_0, and
- there is no line containing p_1, p_2 and p_3

is the indeterminacy set of a Jonquières map of degree 3. More generally on the complement of a strict Zariski closed subset of $\mathcal{J}_d$ the points p_0, p_1, $\ldots$, p_{2d-2} form a set of $2d-1$ distinct points in the complex projective plane. Hence the base points of a generic element ϕ of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})\times\mathcal{J}_d\times\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ are $p_0 = (1:0:0)$ and $2d-1$ distinct points p_1, p_2, $\ldots$, p_{2d-2} of $\mathbb{P}^2_\mathbb{C}$.

Determine $\phi_\bullet(h)$.

3.3 Some applications

3.3.1 Tits alternative

Linear groups satisfy Tits alternative. Recall that a group G is **solvable** if there exists an integer k such that $G^{(k)} = \{\mathrm{id}\}$ where $G^{(0)} = G$ and for $k \geq 1$

$$G^{(k)} = [G^{(k-1)}, G^{(k-1)}] = \langle aba^{-1}b^{-1} \,|\, a,\, b \in G^{(k-1)}\rangle.$$

Theorem 3.3 ([35])**.** *Let $\Bbbk$ be a field of characteristic 0, and Γ be a finitely generated subgroup of $\mathrm{GL}(n,\Bbbk)$. Then*

- *either* Γ *contains a non abelian, free group;*
- *or* Γ *contains a solvable subgroup of finite index.*

The group of diffeomorphisms of a real manifold of dimension ≥ 1 does not satisfy Tits alternative ([22]). The group of polynomial automorphisms of $\mathbb{C}^2$ satisfies Tits alternative ([29]); to prove it Lamy uses the structure of amalgamated product of $\mathrm{Aut}(\mathbb{C}^2)$ that implies that $\mathrm{Aut}(\mathbb{C}^2)$ acts on a tree ([34]). Using the action of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ on $\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ Cantat studied the finitely generated subgroups of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ and establishes the following statement

Theorem 3.4 ([7])**.** *The Cremona group* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *satisfies Tits alternative.*

3.3.2 Simplicity

Let us recall that a group is **simple** if it has no non trivial, proper and normal subgroup. The **normal subgroup of G generated by** $f \in G$ is

$$\langle hfh^{-1} \,|\, h \in G\rangle.$$

Remark that $\mathrm{Aut}(\mathbb{C}^2)$ is not simple: let Ψ be the morphism defined by

$$\mathrm{Aut}(\mathbb{C}^2) \to \mathbb{C}^* \qquad \phi \mapsto \det \mathrm{jac}\, \phi$$

its kernel is a proper normal subgroup of $\mathrm{Aut}(\mathbb{C}^2)$. Danilov has established that $\ker \Psi$ is not simple ([13]); more precisely using [33] he proved that the normal subgroup generated by $(ea)^{13}$ where

$$a = (y, -x) \qquad\qquad e = (x, y + 3x^5 - 5x^4)$$

is a strict subgroup of $\big\{\phi \in \mathrm{Aut}(\mathbb{C}^2) \,|\, \Psi(\phi) = 1\big\}$. More recently Furter and Lamy gave a more precise statement ([21]).

What about the Cremona group ? A birational map ϕ is **tight** if

- $\phi_\bullet$ is hyperbolic;
- there exists a positive number ε such that: if ψ is a birational map, and if $\psi_\bullet(\mathrm{Ax}(\phi_\bullet))$ contains two points at distance ε which are at distance at most 1 from $\mathrm{Ax}(\phi_\bullet)$ then $\psi_\bullet(\mathrm{Ax}(\phi_\bullet)) = \mathrm{Ax}(\phi_\bullet)$;
- if ψ is a birational map and $\psi_\bullet(\mathrm{Ax}(\phi_\bullet)) = \mathrm{Ax}(\phi_\bullet)$, then $\psi\phi\psi^{-1} = \phi^{\pm 1}$.

Using the action of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ on $\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ Cantat and Lamy proved that:

Theorem 3.5 ([8])**.** *Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. If ϕ is tight, then ϕ^k generates a non trivial, strict and normal subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ for some positive integer k.*

As a consequence:

Corollary 3.6 ([8])**.** *The Cremona group $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ contains an uncountable number of strict normal subgroups.*

In particular $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is not simple.

3.3.3 Homomorphisms from lattices into the Cremona group

Using the embedding of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ into the Picard-Manin space, Cantat proved the following result:

Theorem 3.7 ([7])**.** *Any homomorphism with infinite image from a discrete Kazhdan group into $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is conjugate to a homomorphism into $\mathrm{PGL}(3,\mathbb{C})$.*

In particular, this result applies to any lattice Γ in a connected simple Lie group with property (T)[3] but left open the problem of classifying homomorphisms from lattices in the groups $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ into $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. There exist, for some values of n, injective homomorphisms from lattices in $\mathrm{SO}(n,1)$ to the Cremona group ([8, 19]). Delzant and Py focus on the case $\mathrm{SU}(n,1)$:

Theorem 3.8 ([15])**.** *Let Γ be a cocompact lattice in the group $\mathrm{SU}(1,n)$ with $n\geq 2$. If $\rho\colon\Gamma\to\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is an injective homomorphism, then one of the following two possibilities holds*

- *the group $\rho(\Gamma)$ fixes a point in the Picard-Manin space;*
- *the group $\rho(\Gamma)$ fixes a unique point in the boundary of the Picard-Manin space.*

3.3.4 Solvable subgroups

The study of solvable groups started a long time ago, and any linear solvable subgroup is up to finite index triangularizable (Lie-Kolchin theorem, [28, Theorem 21.1.5]). The assumption "up to finite index" is essential: for instance the subgroup of $\mathrm{PGL}(2,\mathbb{C})$ generated by the matrices

$$\begin{bmatrix} 1 & 0\\ 1 & -1\end{bmatrix}\qquad\begin{bmatrix} -1 & 1\\ 0 & 1\end{bmatrix}$$

is isomorphic to $\mathfrak{S}_3$ so is solvable but is not triangularizable.

[3]Informally, a locally compact topological group G has property (T) if it satisfies the following property: if G acts unitarily on a Hilbert space and has "almost invariant vectors", then it has a nonzero invariant vector.

Theorem 3.9 ([16]). *Let G be an infinite, solvable, non virtually abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. *Then, up to finite index, one of the following holds*

1. *any element of G is either of finite order, or conjugate to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$;*

2. *G preserves a unique fibration that is rational, in particular G is, up to conjugacy, a subgroup of* $\mathrm{PGL}(2,\mathbb{C}(y)) \rtimes \mathrm{PGL}(2,\mathbb{C})$*;*

3. *G preserves a unique fibration that is elliptic;*

4. *G is, up to birational conjugacy, a subgroup of*

$$\left\{(x^py^q, x^ry^s),\, (\alpha x, \beta y) \,|\, \alpha,\, \beta \in \mathbb{C}^*\right\}$$

where $M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ *denotes an element of* $\mathrm{GL}(2,\mathbb{Z})$ *with spectral radius* > 1. *The group G preserves the two holomorphic foliations defined respectively by the 1-forms*

$$\alpha_1\, x\mathrm{d}y + \beta_1\, y\mathrm{d}x \qquad \alpha_2\, x\mathrm{d}y + \beta_2\, y\mathrm{d}x$$

where (α_1,β_1) *and* (α_2,β_2) *denote the eigenvectors of* ${}^tM$.

Furthermore if G is uncountable, case 3. does not hold.

Examples 3.10.

- Denote by S_3 the group generated by the matrices

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

As we recall before $\mathrm{S}_3 \simeq \mathfrak{S}_3$. Consider now the subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ whose elements are the monomial maps (x^py^q, x^ry^s) with $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathrm{S}_3$. Then any element of G has finite order, and G is solvable; it gives an example of case 1.

- The centralizer of a birational map of $\mathbb{P}^2_{\mathbb{C}}$ that preserves a unique fibration that is rational is virtually solvable ([10, Corollary C]); this example falls in case 2 (we will give some details in Example 3.17).

- In [12, Proposition 2.2] Cornulier proved that the group

$$\langle (x+1,y),\, (x,y+1),\, (x,xy)\rangle$$

is solvable of length 3, and is not linear over any field; this example falls in case 2. The invariant fibration is given by $x = \mathrm{cst}$.

Exercice 30. Give a subgroup of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ that illustrates case 1.

Exercice 31. Give a subgroup of $\mathrm{Aut}(\mathbb{C}^2)$ that illustrates case 1.

Remark 3.11. In case 1. if there exists an integer d such that $\deg\phi \leq d$ for any ϕ in G, then there exists a birational map $\psi\colon M \dashrightarrow \mathbb{P}^2_\mathbb{C}$ such that $\psi^{-1}G\psi$ is a solvable subgroup of $\mathrm{Aut}(M)$ (see the end of the section for more details). But there is some solvable subgroups G with only elliptic elements that do not satisfy this property: the group

$$\mathrm{E} = \big\{(\alpha x + P(y), \beta y + \gamma) \,|\, \alpha,\, \beta \in \mathbb{C}^*,\, \gamma \in \mathbb{C},\, P \in \mathbb{C}[y]\big\} \subset \mathrm{Aut}(\mathbb{C}^2).$$

We will prove Theorem 3.9: we first assume that our solvable, infinite and non virtually abelian, subgroup G contains a hyperbolic map, then that it contains a twist and no hyperbolic map, and finally that all elements of G are elliptic.

A. Solvable groups of birational maps containing a hyperbolic map

Let us recall the following criterion (for its proof see for example [14]) used on many occasions by Klein, and also by Tits ([35]):

Lemma 3.12 (Ping-Pong Lemma)**.** *Let* H *be a group acting on a set* X*, let* Γ_1*,* Γ_2 *be two subgroups of* H*, and let* Γ *be the subgroup generated by* Γ_1 *and* Γ_2*. Assume that* Γ_1 *contains at least three elements, and* Γ_2 *at least two elements. Suppose that there exist two non-empty subsets* X_1*,* X_2 *of* X *such that* $X_2 \not\subset X_1$*, and for any* $\gamma \in \Gamma_1 \smallsetminus \{\mathrm{id}\}$ *and any* $\gamma' \in \Gamma_2 \smallsetminus \{\mathrm{id}\}$

$$\gamma(X_2) \subset X_1 \qquad \gamma'(X_1) \subset X_2.$$

Then Γ *is isomorphic to the free product* $\Gamma_1 * \Gamma_2$*.*

The Ping-Pong argument allows us to prove the following:

Lemma 3.13 ([16])**.** *A solvable, non abelian, subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *cannot contain two hyperbolic maps* ϕ *and* ψ *such that* $\big\{\omega(\phi_\bullet), \alpha(\phi_\bullet)\big\} \neq \big\{\omega(\psi_\bullet), \alpha(\psi_\bullet)\big\}$*.*

Proof. Assume by contradiction that $\big\{\omega(\phi_\bullet), \alpha(\phi_\bullet)\big\} \neq \big\{\omega(\psi_\bullet), \alpha(\psi_\bullet)\big\}$. Then the Ping-Pong argument implies that there exist two integers n and m such that ψ^n and ϕ^m generate a subgroup of G isomorphic to the free group F_2 (*see* [7]). But $\langle\phi, \psi\rangle$ is a solvable group: contradiction. □

Let G be an infinite solvable, non virtually abelian, subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Assume that G contains a hyperbolic map ϕ. Let $\alpha(\phi_\bullet)$ and $\omega(\phi_\bullet)$ be the two fixed points of $\phi_\bullet$ on $\partial\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$, and $\mathrm{Ax}(\phi_\bullet)$ be the geodesic passing through these two points. As G is solvable there exists a subgroup of G of index 2 that preserves $\alpha(\phi_\bullet)$, $\omega(\phi_\bullet)$, and $\mathrm{Ax}(\phi_\bullet)$ (*see* [7, Theorem 6.4]); let us still denote by G this subgroup. One thus has a morphism $\kappa\colon G\to\mathbb{R}_+^*$ such that

$$\psi_\bullet(\ell)=\kappa(\psi)\ell$$

for any ℓ in $\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ lying on $\mathrm{Ax}(\phi_\bullet)$.

Gap property:

If $\phi\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is a hyperbolic map, then $\lambda(\phi)$ is an algebraic integer with all Galois conjugates in the unit disk, that is a Salem number, or a Pisot number. The smallest known number is the Lehmer number $\lambda_L\simeq 1,176$ which is a root of

$$X^{10}+X^9-X^7-X^6-X^5-X^4-X^3+X+1.$$

Blanc and Cantat prove in [3, Corollary 2.7] that there is a gap in the dynamical spectrum $\Lambda=\big\{\lambda(\phi)\,|\,\phi\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})\big\}$: there is no dynamical degree in $]1,\lambda_L[$.

The gap property implies that in fact $\kappa\colon\psi\to\kappa(\psi)$ such that $\psi_\bullet(\ell)=\kappa(\psi)\ell$ for any ℓ in $\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ lying on $\mathrm{Ax}(\phi_\bullet)$ is a morphism from G to $\mathbb{Z}$. Furthermore $\ker\kappa$ is an infinite subgroup that contains only elliptic maps. Indeed it is clear that the set of elliptic elements of G coincides with $\ker\alpha$; and $[G,G]\subset\ker\alpha$ so if $\ker\alpha$ is finite, G is abelian up to finite index which is impossible.

Elliptic subgroups of the Cremona group with a large normalizer:

Consider in $\mathbb{P}^2_\mathbb{C}$ the complement of the union of the three lines $\{x=0\}$, $\{y=0\}$ and $\{z=0\}$. Denote by $\mathcal{U}$ this open set isomorphic to $\mathbb{C}^*\times\mathbb{C}^*$. One has an action of $\mathbb{C}^*\times\mathbb{C}^*$ on $\mathcal{U}$ by translation. Furthermore $\mathrm{GL}(2,\mathbb{Z})$ acts on $\mathcal{U}$ by monomial maps

$$\left[\begin{array}{cc} p & q\\ r & s\end{array}\right]\mapsto\big((x,y)\mapsto(x^py^q,x^ry^s)\big)$$

One thus has an injective morphism from $(\mathbb{C}^*\times\mathbb{C}^*)\rtimes\mathrm{GL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Let G_{toric} be its image.

One can now apply [15, Theorem 4] that says that if there exists a short exact sequence

$$1\longrightarrow \mathrm{A}\longrightarrow \mathrm{N}\longrightarrow \mathrm{B}\longrightarrow 1$$

where $\mathrm{N}\subset\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ contains at least one hyperbolic element, and $\mathrm{A}\subset\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is an infinite and that fixes a point in $\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$, then N is up to conjugacy a subgroup of G_{toric}. Hence up to birational conjugacy $G\subset G_{\text{toric}}$.

One can now state:

Proposition 3.14 ([16]). *Let G be an infinite solvable, non virtually abelian, subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. *If G contains a hyperbolic birational map, then G is, up to conjugacy and finite index, a subgroup of*

$$\langle (x^py^q, x^ry^s),\, (\alpha x, \beta y)\,|\,\alpha,\, \beta \in \mathbb{C}^*\rangle$$

where $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ *denotes an element of* $\mathrm{GL}(2,\mathbb{Z})$ *with spectral radius* > 1.

B. Solvable groups with a twist

Consider a solvable, non abelian, subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Let us assume that G contains a twist ϕ; the map ϕ preserves a unique fibration $\mathcal{F}$ that is rational or elliptic. Let us prove that any element of G preserves $\mathcal{F}$. Denote by $\alpha(\phi_\bullet) \in \partial\mathbb{H}_{\mathbb{P}^2_{\mathbb{C}}}$ the fixed point of $\phi_\bullet$. Take one element in $\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}}$ still denoted $\alpha(\phi_\bullet)$ that represents $\alpha(\phi_\bullet)$. Take $\varphi \in G$ such that $\varphi(\alpha(\phi_\bullet)) \neq \alpha(\phi_\bullet)$. Then $\psi = \varphi\phi\varphi^{-1}$ is parabolic and fixes the unique element $\alpha(\psi_\bullet)$ of $\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}}$ proportional to $\varphi(\alpha(\phi_\bullet))$. Take $\varepsilon > 0$ such that $\mathcal{U}(\alpha(\phi_\bullet),\varepsilon) \cap \mathcal{U}(\alpha(\psi_\bullet),\varepsilon) = \emptyset$ where

$$\mathcal{U}(\alpha,\varepsilon) = \big\{\ell \in \mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}}\,|\,\alpha\cdot\ell < \varepsilon\big\}.$$

Since $\psi_\bullet$ is parabolic, then for n large enough the inclusion

$$\psi_\bullet^n\big(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon)\big) \subset \mathcal{U}(\alpha(\psi_\bullet),\varepsilon)$$

holds. For m sufficiently large

$$\phi_\bullet^m\psi_\bullet^n\big(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon)\big) \subset \big(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon/2)\big) \subsetneq \big(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon)\big).$$

Hence $\phi_\bullet^m\psi_\bullet^n$ is hyperbolic. You can by this way build two hyperbolic maps whose sets of fixed points are distinct: this gives a contradiction with Lemma 3.13. So for any $\varphi \in G$ one has : $\alpha(\phi_\bullet) = \alpha(\varphi_\bullet)$; one can thus state the following result.

Proposition 3.15 ([16]). *Let G be a solvable, non abelian, subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *that contains a twist ϕ. Then*

- *if ϕ is a Jonquières twist, then G preserves a rational fibration, that is up to birational conjugacy G is a subgroup of* $\mathrm{PGL}(2,\mathbb{C}(y)) \rtimes \mathrm{PGL}(2,\mathbb{C})$,
- *if ϕ is a Halphen twist, then G preserves an elliptic fibration.*

If G is uncountable, then ϕ is a Jonquières twist.

Remark 3.16. Both cases are mutually exclusive.

Example 3.17. If $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ preserves a unique fibration that is rational then one can assume that up to birational conjugacy this fibration is given, in the affine chart $z = 1$, by $y = \mathrm{cst}$. If ϕ preserves $y = \mathrm{cst}$ fiberwise, then

- ϕ is contained in a maximal abelian subgroup denoted $\mathrm{Ab}(\phi)$ that preserves $y = \mathrm{cst}$ fiberwise ([17]),
- the centralizer of ϕ is a finite extension of $\mathrm{Ab}(\phi)$ (*see* [10, Theorem B]).

This allows us to establish that if ϕ preserves a fibration not fiberwise, then the centralizer of ϕ is virtually solvable ([10, Corollary C]). For instance if $\phi = \big(x + a(y), y + 1\big)$ (resp. $\big(b(y)x, \beta y\big)$ or $\big(x + a(y), \beta y\big)$ with $\beta \in \mathbb{C}^*$ of infinite order) preserves a unique fibration, then the centralizer of ϕ is solvable and metabelian ([10, Propositions 5.1 and 5.2]).

C. Solvable groups with no hyperbolic map, and no twist

Let M be a smooth, irreducible, complex, projective variety of dimension n. Fix a Kähler form ω on M. If ℓ is a positive integer, denote by $x_i \colon M^\ell \to M$ the projection onto the i-th factor. The manifold M^ℓ is then endowed with the Kähler form $\displaystyle\sum_{i=1}^{\ell} x_i^*\omega$ which induces a Kähler metric. To any $\phi \in \mathrm{Bir}(M)$ one can associate its graph $\Gamma_\phi \subset M \times M$ defined as the Zariski closure of

$$\big\{(z, \phi(z)) \in M \times M \,|\, z \in M \smallsetminus \mathrm{Ind}\,\phi\big\}.$$

By construction Γ_ϕ is an irreducible subvariety of $M \times M$ of dimension n. Both projections x_1, $x_2 \colon M \times M \to M$ restrict to birational morphisms $x_1, x_2 \colon \Gamma_\phi \to M$.

The **total degree** $\mathrm{tdeg}\,\phi$ of $\phi \in \mathrm{Bir}(M)$ is defined as the volume of Γ_ϕ with respect to the fixed metric on $M \times M$:

$$\mathrm{tdeg}\,\phi = \int_{\Gamma_\phi} \big(x_1^*\omega + x_2^*\omega\big)^n = \int_{M \smallsetminus \mathrm{Ind}\,\phi} \big(\omega + \phi^*\omega\big)^n.$$

Let $d \geq 1$ be a natural integer, and set

$$\mathrm{Bir}_d(M) = \big\{\phi \in \mathrm{Bir}(M) \,|\, \mathrm{tdeg}\,\phi \leq d\big\}.$$

A subgroup G of $\mathrm{Bir}(M)$ has **bounded degree** if it is contained in $\mathrm{Bir}_d(M)$ for some $d \in \mathbb{N}^*$.

Any subgroup G of $\mathrm{Bir}(M)$ that has bounded degree can be regularized, that is up to birational conjugacy all indeterminacy points of all elements of G disappear simultaneously:

Theorem 3.18 ([36]). *Let M be a complex projective variety, and let G be a subgroup of* $\mathrm{Bir}(M)$. *If G has bounded degree, there exists a smooth, complex, projective variety M', and a birational map $\psi \colon M' \dashrightarrow M$ such that $\psi^{-1}G\psi$ is a subgroup of* $\mathrm{Aut}(M')$.

Solution 26. — Let p be a point of $\mathbb{P}^2_\mathbb{C}$, let S_1 be the surface obtained by blowing up $\mathbb{P}^2_\mathbb{C}$ at p, and let E_p be the exceptional divisor of this blow-up. Consider a point q on E_p; denote by S_2 the surface obtained by blowing up q and by E_q the associated exceptional divisor. Both e_p and e_q belong to the image of $\mathrm{NS}(S_2)$ in $\mathcal{Z}_{\mathbb{P}^2_\mathbb{C}}$. Let $\widetilde{E}_p$ be the strict transform of E_p in S_2. Then e_p corresponds to $\widetilde{E}_p+E_q$ and e_q to E_q. Hence

$$\begin{cases} e_p\cdot e_p=\widetilde{E}_p^2+E_q^2+2\widetilde{E}_p\cdot E_q=-2-1+2=-1\\ e_p\cdot e_q=(\widetilde{E}_p\cdot E_q)+E_q^2=1-1=0 \text{ if } p\neq q\end{cases}$$

Solution 27. — As $p=\alpha(\phi_\bullet)+\omega(\phi_\bullet)\in\mathrm{Ax}(\phi_\bullet)$ then

$$\phi_\bullet(p)=\frac{\alpha(\phi_\bullet)}{\lambda(\phi)}+\lambda(\phi)\omega(\phi_\bullet).$$

Since $\phi_\bullet\big(\alpha(\phi_\bullet)\big)=\frac{\alpha(\phi_\bullet)}{\lambda(\phi)}$ and $\phi_\bullet\big(\omega(\phi_\bullet)\big)=\lambda(\phi)\omega(\phi_\bullet)$ we have:

$$2\cosh(\mathrm{dist}(p,\phi_\bullet(p)))=2p\cdot\phi_\bullet(p)=\lambda(\phi)+\frac{1}{\lambda(\phi)}.$$

Furthermore

$$2\cosh(L(\phi_\bullet))=e^{L(\phi_\bullet)}+\frac{1}{e^{L(\phi_\bullet)}}$$

Solution 28. — If ϕ is an isomorphism on a neighborhood of p, and $\phi(p)=q$, then $\phi_\bullet(e_p)=e_q$.

If $L_{q_1q_2}$ is blown down onto p_0 by ϕ^{-1}, then

$$\phi_\bullet(e_{p_0})=h-e_{q_1}-e_{q_2}\qquad \phi_\bullet(h)=2h-e_{q_0}-e_{q_1}-e_{q_2}$$

where h is the class of a line in $\mathbb{P}^2_\mathbb{C}$.

Solution 29. — One has

$$\phi_\bullet(h)=dh-(d-1)e_{p_0}-\sum_{i=1}^{2d-2}e_{p_i}$$

where the p_i's are generic distinct points of $\mathbb{P}^2_\mathbb{C}$.

Solution 30. A subgroup of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ that illustrates case 1. is

$$\big\{(\alpha x+\beta y+\gamma,\delta y+\varepsilon)\,|\,\alpha,\,\delta\in\mathbb{C}^*,\,\beta,\,\gamma,\,\varepsilon\in\mathbb{C}\big\}\subset\mathrm{Aut}(\mathbb{P}^2_\mathbb{C}).$$

Solution 31. A subgroup of $\mathrm{Aut}(\mathbb{C}^2)$ that illustrates case 1. is

$$\mathrm{E}=\big\{(\alpha x+P(y),\beta y+\gamma)\,|\,\alpha,\,\beta\in\mathbb{C}^*,\,\gamma\in\mathbb{C},\,P\in\mathbb{C}[y]\big\}\subset\mathrm{Aut}(\mathbb{C}^2).$$

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