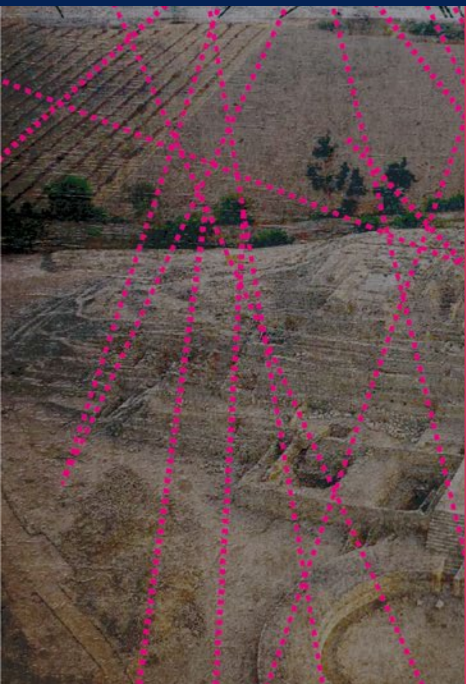
Capítulo 4



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Algebraic properties of groups of complex analytic local diffeomorphisms

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Contents

1 Introduction

We study in these notes the algebraic properties of groups of holomorphic local diffeomorphisms. In this spirit we introduce the basic notions of the theory of pro-algebraic groups. Pro-algebraic groups are the analogue of algebraic linear groups in the infinite dimensional setting of groups of holomorphic local diffeomorphisms. They are very useful to study properties that determine groups defined by algebraic equations in every space of jets.

Let us indicate some examples of the study of algebraic group properties that can be found in the literature. The first example is the study of integrability properties of holomorphic foliations. Given a holomorphic foliation and a leaf we obtain a holonomy group as an image of a representation of the fundamental group of a leaf. It is possible to relate the properties of the derived series of these groups with existence of first integrals or integrating factors. Initially this point of view was developed to study codimension 1 foliations [15, Mattei-Moussu], [16, Paul]... and it has been applied more recently to one-dimensional foliations [18, Rebelo-Reis] [4, Câmara-Scardua]...

Another example is provided by groups of real analytic diffeomorphisms of compact surfaces. The properties of groups of local diffeomorphisms are crucial to show that any nilpotent group of real analytic diffeomorphisms of the sphere is always metabelian, i.e. its first derived group is abelian [7, Ghys]. It is interesting that algebraic properties can be exploited to deduce dynamical properties of groups [19, Rebelo-Reis] [20]. Other applications of the algebraic techniques are the study of the existence of faithful analytic actions of mapping class groups of surfaces on surfaces [5, Cantat-Cerveau], local intersection dynamics [23, Seigal-Yakovenko] [2, Binyamini], derived length [13, Martelo-Ribón] [21]...

We try to give a glimpse of the power of the theory of pro-algebraic groups of formal diffeomorphisms. We lay the groundwork for the study of the derived length of solvable subgroups of local diffeomorphisms in section 4. The fruits of this approach are the sharp bounds for the derived length presented in the results at the end of section 4. We did not prove such theorems in order to keep the text as elementary as possible. The text contains other examples of the utility of pro-algebraic groups in sections 3.6, 3.7, 3.8 and 3.9 that hopefully will motivate the reader. Sometimes we provide a more conceptual interpretation of well-known properties. But we also give very simple proofs of sophisticated results. For instance we show that a group of local diffeomorphisms in dimension n whose elements leave invariant n independent first integrals is necessarily finite (cf. Proposition 3.46).

Another good application is the uniform bound of the period of periodic analytic curves, i.e. invariant by an iterate of a fixed local diffeomorphism.

Let us outline the notes. Section 2 is devoted to explain basic properties of linear groups and to make the reader familiar with these concepts before generalizing them in the setting of local diffeomorphisms. We introduce pro-algebraic groups, explain their properties and how to find examples in section 3. We study the properties of the derived series of a group of local diffeomorphisms in section 4. We define an analogue of the derived series that is more suitable for algebraic groups than the derived series itself and study the properties of a group in terms of its Lie algebra. We classify the pro-algebraic subgroups in dimension 1 modulo formal conjugacy in section 5. Finally we provide an example of a pathological phenomenon of pro-algebraic groups in section 5.

2 Linear groups

We study the algebraic structure of groups of local complex analytic diffeomorphisms. Our point of view involves applying techniques of linear algebraic groups to obtain analogues for groups of local diffeomorphisms. In the next section we introduce linear algebraic groups and stress some properties that will be revisited later on in the context of diffeomorphisms.

Let us consider subgroups of the linear group $\mathrm{GL}(n,\mathbb{C})$. The elements of $\mathrm{GL}(n,\mathbb{C})$ can be considered as points in $\mathbb{C}^{n^2}$ by identifying each matrix with its list of coefficients. In this way it makes sense to consider the algebraic closure $\overline{G}^z$ of a subgroup G of $\mathrm{GL}(n,\mathbb{C})$. The z superindex stands for Zariski-closure.

Proposition 2.1. *Let G be a subgroup of* $\mathrm{GL}(n,\mathbb{C})$*. The algebraic closure $\overline{G}^z$ of G in* $\mathrm{GL}(n,\mathbb{C})$ *is a group.*

A proof of this result can be found in [3, Proposition 1.3(b), p. 47].

Let us calculate some examples so that we get familiarized with the algebraic closure. It is natural to start our study with cyclic groups.

Definition 2.2. Let $A \in \mathrm{GL}(n,\mathbb{C})$. We say that A is *unipotent* if $A - Id$ is nilpotent or equivalently if $\mathrm{spec}(A) = \{1\}$.

2.1 Example

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us denote by B the matrix in the right hand side so that we have $A = \exp(B)$. The matrix B is nilpotent whereas A is unipotent. Let us calculate the one parameter group $\{\exp(tB) : t \in \mathbb{C}\}$. We have

$$\exp(tB) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 \\ \frac{t^2-t}{2} & t & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly $\{\exp(tB) : t \in \mathbb{C}\}$ is an algebraic group given by

$$a_{jj} = 1 \text{ for all } j \in \{1, \ldots, 6\}, \ a_{21} = a_{32} = a_{54}, \ a_{31} = \frac{a_{21}^2 - a_{21}}{2}$$

and $a_{ij} = 0$ for any coefficient that does not appear in the previous equations.

Since $\exp((s+t)B) = \exp(sB)\exp(tB)$ for $s, t \in \mathbb{C}$, we deduce that $A^k = \exp(kB)$ for any $k \in \mathbb{Z}$. Let P be a polynomial on the coefficients of the matrices of $\mathrm{GL}(n, \mathbb{C})$ that vanishes on the elements of the cyclic group $\langle A \rangle$. The expression $Q(t) := P(\exp(tB))$ is polynomial in t. Moreover it vanishes for $t \in \mathbb{Z}$ since $P(A^k) = 0$ for $k \in \mathbb{Z}$. A complex polynomial that vanishes on the integer numbers is necessarily 0. Thus P vanishes on $\{\exp(tB) : t \in \mathbb{C}\}$. We deduce

$$\langle A \rangle \subset \{\exp(tB) : t \in \mathbb{C}\} \subset \overline{\langle A \rangle}^{z}$$

and then $\overline{\langle A \rangle}^{z} = \{\exp(tB) : t \in \mathbb{C}\}$ since $\{\exp(tB) : t \in \mathbb{C}\}$ is algebraic.

2.2 The closure of the group generated by a unipotent matrix

Let us generalize the previous example. Given a unipotent matrix $A \in \mathrm{GL}(n, \mathbb{C})$ we consider the unique nilpotent matrix B such that $A = \exp(B)$.

How to calculate B?

- We can write A in Jordan normal form and then to obtain B by using indeterminate coefficients.
- Alternatively the formula $\log(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j$ motivates us to define $B = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (A - Id)^j$.

The sum defining B has only finitely many non-vanishing terms since $A - Id$ is nilpotent.

The 1-dimensional complex vector space generated by B is the Lie algebra of the group $\overline{\langle A \rangle}^z$.

Definition 2.3. We denote $\log A = B$. We say that $\log A$ is the *infinitesimal generator* of A. We denote $A^t = \exp(tB)$ for $t \in \mathbb{C}$.

The group $\{A^t : t \in \mathbb{C}\}$ is algebraic. The proof is similar as in the example since we can write B in Jordan normal form. The same argument of the example shows that any polynomial on the coefficients of $\mathrm{GL}(n, \mathbb{C})$ vanishing on $\langle A \rangle$ also vanishes on $\{A^t : t \in \mathbb{C}\}$. As a consequence we obtain

Proposition 2.4. *Let A be a unipotent element of $\mathrm{GL}(n, \mathbb{C})$. Then $\overline{\langle A \rangle}^z$ is equal to* $\{A^t : t \in \mathbb{C}\}$

2.3 The closure of the group generated by a semisimple matrix

Let us consider a diagonal matrix $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathrm{GL}(n, \mathbb{C})$. The algebraic closure $\overline{\langle A \rangle}^z$ is contained in the algebraic group of diagonal matrices. Let us calculate $\overline{\langle A \rangle}^z$.

Definition 2.5. Given $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ we associate the morphism of groups

$$\begin{array}{cccc} \chi_{k_1,\ldots,k_n} & : & (\mathbb{C}^*)^n & \to & \mathbb{C}^* \\ & & (\lambda_1, \ldots, \lambda_n) & \mapsto & \lambda_1^{k_1} \ldots \lambda_n^{k_n} \end{array}.$$

We say that $\{\chi_{k_1,\ldots,k_n} : (k_1, \ldots, k_n) \in \mathbb{Z}^n\}$ is the group of *characters* of the complex torus $(\mathbb{C}^*)^n$.

Remark 2.6. The group operation is the multiplication of characters. Indeed the map $(k_1, \ldots, k_n) \mapsto \chi_{k_1,\ldots,k_n}$ is an isomorphism from $\mathbb{Z}^n$ onto the group of characters.

Definition 2.7. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. We define

$$J_{\underline{\lambda}} = \{\underline{k} \in \mathbb{Z}^n : \chi_{\underline{k}}(\underline{\lambda}) = 1\} \text{ and } G_{\underline{\lambda}} = \{\underline{\mu} \in (\mathbb{C}^*)^n : \chi_{\underline{k}}(\underline{\mu}) = 1 \ \text{ for all } \underline{k} \in J_{\underline{\lambda}}\}.$$

Let us calculate the Zariski-closure of the group generated by a diagonal matrix. The arguments are presented in a series of exercises.

Exercise 2.1. Consider a subset J of $\mathbb{Z}^n$. Show that

$$\{(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n : \chi_{k_1,\dots,k_n}(\lambda_1, \dots, \lambda_n) = 1 \text{ for all } (k_1, \dots, k_n) \in J\}$$

is an algebraic group (we are identifying $(\mathbb{C}^*)^n$ with the diagonal matrices). Deduce that $G_{\underline{\lambda}}$ is an algebraic group containing $\underline{\lambda}$.

Exercise 2.2. Let $\mu_1, \dots \mu_p$ pairwise different non-vanishing complex numbers. Suppose that

$$c_1 \mu_1^k + \dots + c_p \mu_p^k = 0$$

for all $k \in \mathbb{Z}$. Show that $c_1 = \dots = c_p = 0$.

Exercise 2.3. Let $P = \sum_{i_1,\dots,i_n} a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial such that $P(\lambda_1^k, \dots, \lambda_n^k) = 0$ for some $(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$ and any $k \in \mathbb{Z}$. We define

$$S = \{\lambda_1^{i_1} \dots \lambda_n^{i_n} : a_{i_1 \dots i_n} \neq 0\}.$$

We write P in the form $\sum_{\mu \in S} \sum_{\lambda_1^{i_1} \dots \lambda_n^{i_n} = \mu} a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$. Show $\sum_{\lambda_1^{i_1} \dots \lambda_n^{i_n} = \mu} a_{i_1 \dots i_n} = 0$ for any $\mu \in S$.

Exercise 2.4. Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$. Consider

$$I(\langle(\lambda_1, \dots, \lambda_n)\rangle) = \{P \in \mathbb{C}[x_1, \dots, x_n] : P(\lambda_1^k, \dots, \lambda_n^k) = 0 \text{ for all } k \in \mathbb{Z}\}$$

and

$$V(I(\langle\underline{\lambda}\rangle)) = \{\underline{\mu} \in (\mathbb{C}^*)^n : P(\underline{\mu}) = 0 \text{ for all } P \in I(\langle\underline{\lambda}\rangle)\}.$$

Show $G_{\underline{\lambda}} = V(I(\langle\underline{\lambda}\rangle))$.

Proposition 2.8. $\overline{\langle\underline{\lambda}\rangle}^z = G_{\lambda}$.

Proposition 2.8 is a consequence of Exercise 2.4.

Corollary 2.9. *Let* $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$ *such that* $\log \lambda_1, \dots, \log \lambda_n$, $2\pi i$ *are* $\mathbb{Q}$-*linearly independent (notice that the condition does not depend on the choice of* $\log \lambda_j$ *for* $1 \leq j \leq n$*). Then* $\overline{\langle\underline{\lambda}\rangle}^z = (\mathbb{C}^*)^n$.

2.4 The closure of a cyclic group

So far we calculated $\overline{\langle A \rangle}^z$ for $A \in \mathrm{GL}(n, \mathbb{C})$ in two cases, namely if A is diagonalizable or if A is unipotent. What happens in the general case? Let us see that it can be reduced to the previous ones.

Let us introduce the so called Jordan multiplicative decomposition, it is a diagonalizable-unipotent decomposition.

Proposition 2.10. *Let $A \in \mathrm{GL}(n, \mathbb{C})$. There exist unique commuting matrices $A_s, A_u \in \mathrm{GL}(n, \mathbb{C})$ such that A_s is diagonalizable, A_u is unipotent and $A = A_s A_u = A_u A_s$.*

The s in the subindex of A_s stands for semisimple. Indeed since A_s is diagonalizable there exists a direct sum $\bigoplus_{j=1}^n V_j$ where V_j is a vector subspace of dimension 1 of eigenvectors of A_s. Each action $(A_s)_{|V_j} : V_j \to V_j$ is simple, meaning that it can not be decomposed anymore or more rigorously that it is irreducible for any $1 \leq j \leq n$. Since V is decomposed into a sum of simple objects for the action of A_s we say that A_s is semisimple. Anyway, we use diagonalizable and semisimple as synonyms.

The proof is an exercise in linear algebra (cf. [3, Corollary 1, p. 81]). The existence of the decomposition is very easy to prove. Given any matrix the Jordan normal form theorem implies that up to linear change of coordinates it can be decomposed in diagonal blocks. For instance a 3×3 block is of the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \lambda^{-1} & 1 & 0 \\ 0 & \lambda^{-1} & 1 \end{pmatrix}.$$

The right hand side is the multiplicative Jordan decomposition of the block. Proceeding analogously for each block we obtain the Jordan decomposition for the initial matrix.

This decomposition is also called Jordan-Chevalley decomposition. It is due to the following result:

Theorem 2.11 (Chevalley, cf. [3, section I.4.4, p. 83])**.** *Let G be a linear algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$. Given any $A \in G$ both the semisimple and the unipotent parts A_s and A_u of A also belong to G.*

This result is extremely important and very useful to calculate invariance groups associated to geometrical actions. Later on we will see some examples in the context of groups of diffeomorphisms.

The Chevalley's theorem implies that $\overline{\langle A \rangle}^z$ contains A_s and A_u and then the group generated by $\overline{\langle A_s \rangle}^z$ and $\overline{\langle A_u \rangle}^z$. Are we missing some elements? The answer is no!

Proposition 2.12. *Let $A \in \mathrm{GL}(n, \mathbb{C})$. Then $\overline{\langle A \rangle}^z$ is equal to the abelian group generated by $\overline{\langle A_s \rangle}^z$ and $\overline{\langle A_u \rangle}^z$. In particular $\overline{\langle A \rangle}^z$ is isomorphic to the direct product $\overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z$.*

How to prove Proposition 2.12? It is known that algebraic properties of groups do not change when considering the algebraic closure, so since $\langle A \rangle$ is abelian the closure $\overline{\langle A \rangle}^z$ is also abelian. Anyway, let us prove such result in order to gain some familiarity with these concepts.

Lemma 2.13. *Let G be a commutative linear algebraic subgroup of* $\mathrm{GL}(n, \mathbb{C})$*. Then $\overline{G}^z$ is commutative.*

Proof. We define

$$Z(G) = \{A \in \mathrm{GL}(n, \mathbb{C}) : AB - BA = 0 \text{ for all } B \in G\}.$$

Clearly $Z(G)$ is a group (the so called centralizer of G) and it is algebraic since fixed $B \in G$ the equation $AB - BA = 0$ is linear in the coefficients of A. Since G is abelian, $Z(G)$ contains G. We obtain $\overline{G}^z \subset \overline{Z(G)}^z = Z(G)$. In particular G is contained in $Z(\overline{G}^z)$ and since the latter group is algebraic we deduce $\overline{G}^z \subset Z(\overline{G}^z)$. The latter property is equivalent to $\overline{G}^z$ being abelian. □

Exercise 2.5. Lemma 2.13 and Chevalley's theorem imply that the group generated by $\overline{\langle A_s \rangle}^z$ and $\overline{\langle A_u \rangle}^z$ is abelian. Show this result without using Chevalley's theorem.

How to find an algebraic group that contains $\overline{\langle A_s \rangle}^z \cup \overline{\langle A_u \rangle}^z$? We can consider a morphism

$$\begin{array}{cccc} \times & : & \overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z & \to & \mathrm{GL}(n, \mathbb{C}) \\ & & (B, C) & \mapsto & BC \end{array}.$$

The group $\overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z$ can be interpreted as a linear matrix group, for instance as a subgroup of $\mathrm{GL}(2n, \mathbb{C})$ making $\overline{\langle A_s \rangle}^z$ (resp. $\overline{\langle A_u \rangle}^z$) act on the first (resp. last) n coordinates. We claim that it is a morphism of groups and an algebraic morphism, i.e. a morphism of algebraic groups. It is clear that $\times$ is an algebraic morphism. Moreover $\times$ is a morphism of groups since the elements of $\overline{\langle A_s \rangle}^z$ commute with the elements of $\overline{\langle A_u \rangle}^z$. Now we can use the following result:

Proposition 2.14 (cf. [3, Corollary 1.4, p. 47])**.** *Let $\alpha : G \to G'$ be a morphism of matrix algebraic groups. Then $\alpha(G)$ is an algebraic group.*

Remark 2.15. Given a subset M of $\mathrm{GL}(n,\mathbb{C})$ we denote by $\mathcal{A}(M)$ the intersection of the algebraic groups containing M. Clearly $\mathcal{A}(M)$ is an algebraic group, the smallest one containing M. Let $\alpha : G \to G'$ be a morphism of matrix algebraic groups and $M \subset G$. We obtain $\alpha(\mathcal{A}(M)) = \mathcal{A}(\alpha(M))$. Indeed $\alpha(\mathcal{A}(M))$ is an algebraic group containing $\alpha(M)$ by Proposition 2.14 and then $\mathcal{A}(\alpha(M)) \subset \alpha(\mathcal{A}(M))$. Moreover since α is continuous in the Zariski topology we obtain that $\alpha^{-1}(\mathcal{A}(\alpha(M)))$ is an algebraic group containing M and then $\mathcal{A}(M)$. We deduce $\alpha(\mathcal{A}(M)) \subset \mathcal{A}(\alpha(M))$.

Proof of Proposition 2.12. This is just a recap of the discussion above. The semisimple and unipotent parts A_s and A_u of A belong to $\overline{\langle A \rangle}^z$ by Chevalley's theorem. Thus $\overline{\langle A_s \rangle}^z$, $\overline{\langle A_u \rangle}^z$ and then $\times(\overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z)$ are contained in $\overline{\langle A \rangle}^z$. Since $\times(\overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z)$ is algebraic and contains the matrix A, we obtain

$$\times(\overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z) = \overline{\langle A \rangle}^z.$$

The map $\times$ is injective. Indeed if $\times(B,C) = BC = Id$ for some $B \in \overline{\langle A_s \rangle}^z$ and $C \in \overline{\langle A_u \rangle}^z$ then BC is a Jordan-Chevalley decomposition of the identity map. Therefore we obtain $B = Id$ and $C = Id$. As a consequence $\overline{\langle A \rangle}^z$ is isomorphic to $\overline{\langle A_s \rangle}^z \times \overline{\langle A_u \rangle}^z$. □

Exercise 2.6. Let G be an abelian subgroup of $\mathrm{GL}(n,\mathbb{C})$. Show

- The set of semisimple elements of G is a group.
- The set of unipotent elements of G is a group.
- Every semisimple element of G commutes with every unipotent element of G.

The group $\overline{\langle A \rangle}^z$ satisfies the conditions of Exercise 2.6 by Proposition 2.12. The goal of the exercise is extending this property to every abelian matrix group.

Remark 2.16. Is there any other distinguished class of groups that satisfies the properties in Exercise 2.6? Nilpotent groups do (Suprunenko and Tyskevic, cf. [27, Theorem 7.11, p. 97]).

2.5 Some elementary properties of linear algebraic groups

Let G be a linear algebraic matrix subgroup of $\mathrm{GL}(n,\mathbb{C})$. We introduce some properties of algebraic matrix groups that generalize in the setting of local diffeomorphisms. By no means the list is supposed to be exhaustive.

Definition 2.17. We denote by G_0 the connected component of the identity transformation Id.

Proposition 2.18 (cf. [3, Chapter I.1, p. 46])**.** *Let G be a linear algebraic matrix subgroup of* $\mathrm{GL}(n, \mathbb{C})$*. Then*

- *G is a smooth manifold.*
- *G_0 is a closed finite index normal subgroup of G.*
- *Every algebraic subgroup of G of finite index contains G_0.*

In particular G_0 is a linear algebraic group.

Remark 2.19. When we use topological terms as "closed" or "connected" in Proposition 2.18 we are referring to the Zariski topology. Anyway, an algebraic set is connected in the Zariski topology if and only if it is connected in the usual topology. This can be deduced from the connectedness in the usual topology of irreducible algebraic sets (cf. [25, Chapter VII.2.2, Theorem 1]).

Definition 2.20. Let G be a linear algebraic group (or a Lie group). We define

$$L(G) = \{A \in \mathrm{End}(\mathbb{C}^n) : \exp(tA) \in G \text{ for all } t \in \mathbb{C}\}.$$

Equivalently $L(G)$ is the tangent space $T_{Id}G$ of G at Id. We say that $L(G)$ is the Lie algebra of the group G.

Exercise 2.7. Show the equivalence between the two definitions of $L(G)$.

The definition is justified by the next result.

Proposition 2.21. *$L(G)$ is a complex Lie algebra where the Lie bracket $[A, B]$ is defined by $AB - BA$.*

The definition implies that the set $\exp(L(G))$ is contained in G_0. Even if these sets can be different we have

Proposition 2.22 (cf. [26, section 8.6, p. 177])**.** *Let G be a linear algebraic group (or more generally a Lie group). Then $G_0 = \langle \exp(L(G)) \rangle$.*

Exercise 2.8. Show that the Lie algebra of the algebraic group

$$\mathrm{SL}(2, \mathbb{C}) = \{A \in \mathrm{GL}(2, \mathbb{C}) : \det A = 1\}$$

is equal to $\mathrm{sl}(2, \mathbb{C}) = \{A \in \mathrm{End}(\mathbb{C}^2) : \mathrm{Tr}(A) = 0\}$.

Exercise 2.9. Show that any matrix in $\exp(\mathrm{sl}(2,\mathbb{C}))$ has Jordan normal form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Show that

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

does not belong to $\exp(\mathrm{sl}(2,\mathbb{C}))$ and that $\exp : \mathrm{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ is not surjective.

Let us applicate the previous definitions to our test examples.

Remark 2.23. Let $A \in \mathrm{GL}(n,\mathbb{C})$ be unipotent. The Lie algebra of $\overline{\langle A \rangle}^{z}$ is the 1-dimensional complex vector space generated by the infinitesimal generator $\log A$ of A. The group $\overline{\langle A \rangle}^{z}$ is connected, indeed it is isomorphic to $\mathbb{C}$.

Exercise 2.10. Let G be a linear algebraic group. Consider a unipotent element A of G. Show that A belongs to G_0.

Exercise 2.11. Consider the diagonal matrix $A = \mathrm{diag}(\underline{\lambda})$ where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. Let $J_{\underline{\lambda}}$ the subgroup of $\mathbb{Z}^n$ of definition 2.7. We define $J'_{\underline{\lambda}}$ as the intersection of the $\mathbb{Q}$-vector space generated by $J_{\underline{\lambda}}$ and $\mathbb{Z}^n$. Show that the group

$$\{\underline{\mu} \in (\mathbb{C}^*)^n : \chi_{k_1,\ldots,k_n}(\underline{\mu}) = 1 \text{ for all } (k_1,\ldots,k_n) \in J'_{\underline{\lambda}}\}$$

is equal to the connected component of the identity of $\overline{\langle A \rangle}^{z}$. Show

$$L(\overline{\langle A \rangle}^{z}) = \{\mathrm{diag}(\mu_1,\ldots,\mu_n) : k_1\mu_1 + \ldots + k_n\mu_n = 0 \text{ for all } (k_1,\ldots,k_n) \in J_{\underline{\lambda}}\}.$$

Let us show a result that will be useful later on. We obtain the Zariski-closure of $\langle A \rangle$ as a Zariski-closure of groups generated by iterates of A.

Proposition 2.24. *Let $A \in \mathrm{GL}(n,\mathbb{C})$. Consider $k \in \mathbb{Z} \setminus \{0\}$ such that $A^k \in \overline{\langle A \rangle}^{z}_0$. Then we obtain $\overline{\langle A^k \rangle}^{z} = \overline{\langle A \rangle}^{z}_0$.*

Proof. We denote $H = \overline{\langle A^k \rangle}^{z}$. Since $A\langle A^k \rangle A^{-1} = \langle A^k \rangle$ we deduce $AHA^{-1} = H$. The group H is a finite index subgroup of $\langle H, A \rangle$. Morever since H is algebraic, the group $\langle H, A \rangle$ is algebraic; indeed $\langle H, A \rangle$ is the algebraic closure of $\langle A \rangle$. The last item of Proposition 2.18 implies $\overline{\langle A \rangle}^{z}_0 \subset H$. Since $H \subset \overline{\langle A \rangle}^{z}_0$ by the choice of k, we obtain $\overline{\langle A^k \rangle}^{z} = \overline{\langle A \rangle}^{z}_0$. □

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{End}(\mathbb{C}^n)$. When is $\mathfrak{g}$ algebraic? More precisely, when is $\mathfrak{g}$ the Lie algebra of an algebraic matrix group? There is a complete answer for this question (cf. [3, Chapter II, section 7]). Let us focus though on a simpler problem in the next exercise.

Exercise 2.12. Let $\underline{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$. Suppose that $\mu_1, \ldots, \mu_n$ are $\mathbb{Q}$-linearly independent. Show that the Lie algebra generated by $\mathrm{diag}(\underline{\mu})$ is non-algebraic.

2.6 Classical results

Let us introduce well-known results by Lie and Kolchin about the structure of groups of unipotent elements and solvable groups.

Theorem 2.25 (Kolchin, cf. [24, chapter V, p. 35])**.** *Let V be a finite dimensional vector space over a field K. Let G be a subgroup of $\mathrm{GL}(V)$ such that each element $g \in G$ is unipotent. Then up to a change of base G is a group of upper triangular matrices.*

Theorem 2.26 (Lie-Kolchin, cf. [8, section 17.6, p. 113])**.** *Let G be a solvable connected subgroup of $\mathrm{GL}(n, F)$ where F is an algebraically closed field. Then up to a change of base G is a group of upper triangular matrices.*

2.7 More properties of algebraic groups

We continue describing the properties of the algebraic closure of a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Definition 2.27. Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$. We denote by G_u the subset of G of unipotent transformations. We say that the group G is *unipotent* if $G = G_u$.

Definition 2.28. Let G be a group. We define the *derived group* $G^{(1)}$ (or $[G, G]$) of G as

$$G^{(1)} = \langle fgf^{-1}g^{-1} : f, g \in G \rangle,$$

i.e. $G^{(1)}$ is the subgroup generated by the commutators of elements of G. We define $G^{(2)} = [G^{(1)}, G^{(1)}]$, $G^{(3)} = [G^{(2)}, G^{(2)}], \ldots$ recursively. We denote $G^{(0)} = G$.

Definition 2.29. We say that G is *solvable* if there exists $p \in \mathbb{N} \cup \{0\}$ such that $G^{(p)} = \{1\}$. Moreover the minimum such p is called the *derived length* $\ell(G)$ of G. We define $\ell(G) = \infty$ if G is non-solvable.

Lemma 2.30. *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$. Then*

- $\ell(\overline{G}^z) = \ell(G)$.
- $(\overline{G}^z)_u \subset (\overline{G}^z)_0$.
- $\overline{G}^z = (\overline{G}^z)_u$ *if G is unipotent.*
- $(\overline{G}^z)_u$ *is a closed normal connected subgroup of the group $\overline{G}^z$ if G is solvable.*

Proof. Since $\ell(G) \leq \ell(\overline{G}^z)$ it suffices to prove that $G^{(p)} = \{Id\}$ implies $(\overline{G}^z)^{(p)} = \{Id\}$. The property $G^{(p)} = \{Id\}$ is equivalent to a system of algebraic equations. The system is also satisfied for $\overline{G}^z$ by definition of the Zariski-closure. Hence we obtain $(\overline{G}^z)^{(p)} = \{Id\}$.

The second item is a consequence of Exercise 2.10.

We claim that $(\overline{G}^z)_u$ is an algebraic subset of $\overline{G}^z$. Indeed it is the subset of $\overline{G}^z$ defined by the equation $(A - Id)^n = 0$ that is algebraic in the coefficients of A. Let G be a unipotent group. We have $G \subset (\overline{G}^z)_u \subset \overline{G}^z$. Since $\overline{G}^z$ is the minimal algebraic set containing G, we deduce $(\overline{G}^z)_u = \overline{G}^z$.

Let us show the last item. We already proved that $(\overline{G}^z)_u$ is closed (or equivalently algebraic). Given $A \in (\overline{G}^z)_u$ we have

$$\{A^t : t \in \mathbb{C}\} = \overline{\langle A \rangle}^z \subset \overline{(\overline{G}^z)_u}^z = (\overline{G}^z)_u$$

and $\{A^t : t \in \mathbb{C}\}$ is a connected set containing Id and A. Therefore $(\overline{G}^z)_u$ coincides with its connected component of Id and it is connected. It is clear that $(\overline{G}^z)_u$ is normal as a set, meaning $A(\overline{G}^z)_u A^{-1} = (\overline{G}^z)_u$ since a conjugate of a unipotent matrix is also unipotent. Notice that we did not use so far that G is solvable, we will use it now to show that $(\overline{G}^z)_u$ is a subgroup.

The group $\overline{G}^z$ is solvable by the first item and $(\overline{G}^z)_0$ is solvable too since it is a subgroup of $\overline{G}^z$. The group $(\overline{G}^z)_0$ is connected by definition, hence we apply Lie-Kolchin's theorem; we can suppose that it is a group of upper triangular matrices up to linear conjugacy. The eigenvalues of an upper triangular matrix are exactly the coefficients in the principal diagonal of the matrix. Thus the elements of $(\overline{G}^z)_u$ are the elements of $(\overline{G}^z)_0$ that have all the elements of the diagonal principal equal to 1. The product of two elements of $(\overline{G}^z)_u$ is still an upper triangular matrix whose principal diagonal coefficients are all equal to 1 and in particular belongs to $(\overline{G}^z)_u$. Analogously the inverse of an element of $(\overline{G}^z)_u$ also belongs to $(\overline{G}^z)_u$. We deduce that $(\overline{G}^z)_u$ is a group. □

Exercise 2.13. Show that the subset of unipotent elements of the algebraic group $\mathrm{GL}(n, \mathbb{C})$ is not a group.

3 Pro-algebraic groups

Inspired by matrix groups we want to define the algebraic closure of a group of local diffeomorphisms. The main problem is that groups of diffeomorphisms can be infinite dimensional. Indeed an element ϕ of $\mathrm{Diff}(\mathbb{C}^n, 0)$ is of the form

$$\phi(x_1, \ldots, x_n) = \Big(\sum_{i_1+\ldots+i_n \geq 1} a^1_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}, \ldots, \sum_{i_1+\ldots+i_n \geq 1} a^n_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} \Big)$$

where the linear part $D_0\phi$ at 0 is an invertible matrix and there are infinitely many coefficients in the power series defining ϕ. Anyway given any degree there are finitely many coefficients up to that degree. This suggests that it could be interesting to truncate a group of diffeomorphisms up to any degree, considering the algebraic closure in each of them and then pasting the information obtained. Let us explain how to execute this strategy in this section.

The first idea is forgetting for a minute that a diffeomorphism is a dynamical object. Let us interpret a diffeomorphism as an operator in a space of functions.

Definition 3.1. We denote by $\hat{\mathcal{O}}_n$ be the local ring $\mathbb{C}[[x_1, \ldots, x_n]]$ of complex formal power series in n variables. We denote by $\mathfrak{m}$ the maximal ideal of $\hat{\mathcal{O}}_n$.

Every local diffeomorphism $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ induces two morphisms of $\mathbb{C}$-algebras by composition in $\hat{\mathcal{O}}_n$ and $\mathfrak{m}$ respectively:

$$\begin{array}{ccc} \hat{\mathcal{O}}_n & \to & \hat{\mathcal{O}}_n \\ f & \mapsto & f \circ \phi \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{m} & \to & \mathfrak{m} \\ f & \mapsto & f \circ \phi. \end{array} \tag{1}$$

The map that associates to any $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ the operator induced by ϕ in $\mathfrak{m}$ or $\hat{\mathcal{O}}_n$ is injective since ϕ is determined by the compositions $x_1 \circ \phi, \ldots, x_n \circ \phi$.

Instead of considering the action of $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ on $\mathfrak{m}$ let us consider the action induced on the finite dimensional vector complex space $\mathfrak{m}/\mathfrak{m}^{k+1}$, i.e. on the space of k-jets. We remind the reader that $\mathfrak{m}^{k+1}$ is the $(k+1)$th-power of the ideal $\mathfrak{m}$. Intuitively we are considering the power series expansion of ϕ up to order k. More precisely we consider the element $\phi_k \in \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ defined by

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^{k+1} & \stackrel{\phi_k}{\to} & \mathfrak{m}/\mathfrak{m}^{k+1} \\ g + \mathfrak{m}^{k+1} & \mapsto & g \circ \phi + \mathfrak{m}^{k+1}. \end{array} \tag{2}$$

Definition 3.2. We define $D_k = \{\varphi_k : \varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)\}$.

Remark 3.3. D_k is a subgroup of the linear group $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$.

Exercise 3.1. Show that D_k is the group of isomorphisms of the $\mathbb{C}$-algebra $\mathfrak{m}/\mathfrak{m}^{k+1}$.

The group D_k can be understood as an algebraic group of matrices by noticing that we have

$$D_k = \{\alpha \in \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{k+1}) : \alpha(gh) = \alpha(g)\alpha(h) \text{ for all } g, h \in \mathfrak{m}/\mathfrak{m}^{k+1}\}$$

and that fixed $g, h \in \mathfrak{m}/\mathfrak{m}^{k+1}$ the equation $\alpha(gh) = \alpha(g)\alpha(h)$ is algebraic on the coefficients of α.

3.1 Example

Let us illustrate the algebraic nature of D_2 for $n = 2$. We denote $x = x_1$ and $y = x_2$. A base of $\mathfrak{m}/\mathfrak{m}^3$ is given by the classes of the monomials of degree 1 and 2, namely x, y, x^2, xy and y^2. Any element A of $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^3)$ is represented by a 5×5 invertible matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

in such a basis. Notice that

$$A(x + \mathfrak{m}^3) = a_{11}x + a_{21}y + a_{31}x^2 + a_{41}xy + a_{51}y^2 + \mathfrak{m}^3$$

and

$$A(y + \mathfrak{m}^3) = a_{12}x + a_{22}y + a_{32}x^2 + a_{42}xy + a_{52}y^2 + \mathfrak{m}^3$$

determine an element A of D_2 since x^2, xy and y^2 are products of x and y. The equation $A(x^2 + \mathfrak{m}^3) = A(x + \mathfrak{m}^3)A(x + \mathfrak{m}^3)$ implies

$$a_{13}x + a_{23}y + a_{33}x^2 + a_{43}xy + a_{53}y^2 = (a_{11}x + a_{21}y + a_{31}x^2 + a_{41}xy + a_{51}y^2)^2$$

modulo $\mathfrak{m}^3$, i.e. modulo discarding the terms of degree greater or equal than 3. In particular we obtain

$$a_{13} = 0,\ a_{23} = 0, a_{33} = a_{11}^2,\ a_{43} = 2a_{11}a_{21},\ a_{53} = a_{21}^2. \tag{3}$$

By analyzing $A(xy + \mathfrak{m}^3) = A(x + \mathfrak{m}^3)A(y + \mathfrak{m}^3)$ and $A(y^2 + \mathfrak{m}^3) = A(y + \mathfrak{m}^3)A(y + \mathfrak{m}^3)$ we obtain

$$a_{14} = 0,\ a_{24} = 0, a_{34} = a_{11}a_{12},\ a_{44} = a_{11}a_{22} + a_{21}a_{12},\ a_{54} = a_{21}a_{22} \tag{4}$$

and

$$a_{15} = 0,\ a_{25} = 0, a_{35} = a_{12}^2,\ a_{45} = 2a_{12}a_{22},\ a_{55} = a_{22}^2 \tag{5}$$

respectively. Equations (3), (4) and (5) determine the algebraic group D_2.

3.2 The group of formal diffeomorphisms

We can think of D_k as the truncation of the group $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ up to the order k. Let us study the relations between the groups D_k for $k \in \mathbb{N}$.

Consider $l \geq k \geq 1$. We want to define a natural map $\pi_{l,k} : D_l \to D_k$ for $l \geq k \geq 1$. The idea is that the truncation of a diffeomorphism up to order l provides all truncations of orders less than l. The map $\pi_{l,k}$ strips the elements of D_l of the information associated to the levels higher than k.

Definition 3.4. Given $l \geq k \geq 1$ and $A \in D_l$ we define $\pi_{l,k}(A)$ as the unique element of D_k such that

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^{l+1} & \xrightarrow{A} & \mathfrak{m}/\mathfrak{m}^{l+1} \\ \downarrow & & \downarrow \\ \mathfrak{m}/\mathfrak{m}^{k+1} & \xrightarrow{\pi_{l,k}(A)} & \mathfrak{m}/\mathfrak{m}^{k+1} \end{array}$$

is commutative where the vertical arrows are the natural projections.

The map $\pi_{l,k} : D_l \to D_k$ is well-defined since every element of D_l leaves invariant every subspace of the form $\mathfrak{m}^p/\mathfrak{m}^{l+1}$ for $1 \leq p \leq l+1$ and in particular $\mathfrak{m}^{k+1}/\mathfrak{m}^{l+1}$.

Exercise 3.2. Let $\phi \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$. Show $\pi_{l,k}(\phi_l) = \phi_k$ for $l \geq k \geq 1$.

Lemma 3.5. *The pair $((D_k)_{k\in\mathbb{N}}, (\pi_{l,k})_{l\geq k\geq 1})$ is an inverse system of algebraic groups and morphisms of algebraic groups. Moreover $\pi_{l,k}$ is surjective for any $l \geq k \geq 1$.*

sketch of proof. It is a simple exercise to check out that $\pi_{l,k}$ is a morphism of algebraic groups. We are just forgetting the action of an element of D_l on $\mathfrak{m}^{k+1}/\mathfrak{m}^{l+1}$.

We have $\pi_{p,p} = Id_{|D_p}$ and $\pi_{j,l} \circ \pi_{l,k} = \pi_{j,k}$ for all $p \in \mathbb{N}$ and $j \geq l \geq k \geq 1$ by Definition 3.4.

Fix $l \geq k \geq 1$. The map $\pi_{l,k}$ is surjective, in fact given $A \in D_k$ there exists by definition $\phi \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\phi_k = A$ and we have $\pi_{l,k}(\phi_l) = \phi_k = A$. $\square$

Definition 3.6. We define the group $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ of formal diffeomorphisms as the projective limit $\varprojlim_{k\in\mathbb{N}} D_k$.

Remark 3.7. Let us remind that the elements of $\varprojlim_{k\in\mathbb{N}} D_k$ are of the form $(A_k)_{k\geq 1}$ where $A_k \in D_k$ and $\pi_{l,k}(A_l) = A_k$ for all $l \geq k \geq 1$. In particular the map

$$\begin{array}{ccc} \mathrm{Diff}\,(\mathbb{C}^n,0) & \to & \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) \\ \phi & \mapsto & (\phi_k)_{k\geq 1} \end{array}$$

is an injective morphism of groups. In this way we see $\mathrm{Diff}\,(\mathbb{C}^n,0)$ as a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$.

Let us give a (maybe) more pleasant presentation of the group of formal diffeomorphism in which it is clear that $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ is the formal completion of $\mathrm{Diff}\,(\mathbb{C}^n,0)$.

We consider the notation $\sum a_{\underline{i}} \underline{x}^{\underline{i}}$ for formal power series where $\underline{i} = (i_1,\ldots,i_n)$ is a multi-index of degree $|\underline{i}| = i_1 + \ldots + i_n$ and $\underline{x}^{\underline{i}} = x_1^{i_1}\ldots x_n^{i_n}$. Given a power series $\sum a_{\underline{i}} \underline{x}^{\underline{i}}$ we define $j^k(\sum a_{\underline{i}} \underline{x}^{\underline{i}}) = \sum_{|\underline{i}|\leq k} a_{\underline{i}} \underline{x}^{\underline{i}}$. We define $j^k(f_1,\ldots,f_n) = (j^k f_1,\ldots,j^k f_n)$ for a n-uple of power series.

Consider the set $\overline{\mathrm{Diff}}(\mathbb{C}^n,0)$ of elements $(\phi_1,\ldots,\phi_n)$ of $\mathfrak{m}^n$ such that $(j^1\phi_1,\ldots,j^1\phi_n)$ is an invertible linar map. The set of elements of $\overline{\mathrm{Diff}}(\mathbb{C}^n,0)$ such that all their coordinates are convergent power series coincides with the group $\mathrm{Diff}\,(\mathbb{C}^n,0)$ by the inverse function theorem. It would be natural to define $\overline{\mathrm{Diff}}(\mathbb{C}^n,0)$ as the group of formal diffeomorphisms too. This is not an issue in our approach since $\varprojlim_{k\in\mathbb{N}} D_k$ and $\overline{\mathrm{Diff}}(\mathbb{C}^n,0)$ can be identified.

Exercise 3.3. Define a group operation in $\overline{\mathrm{Diff}}(\mathbb{C}^n,0)$ such that $\mathrm{Diff}\,(\mathbb{C}^n,0)$ is a subgroup of $\overline{\mathrm{Diff}}(\mathbb{C}^n,0)$.

Given

$$\overline{\eta} = \left(\sum_{|\underline{i}|\geq 1} a_{\underline{i}}^1 \underline{x}^{\underline{i}}, \ldots, \sum_{|\underline{i}|\geq 1} a_{\underline{i}}^n \underline{x}^{\underline{i}}\right) \in \overline{\mathrm{Diff}}(\mathbb{C}^n,0)$$

let us construct an element of $\varprojlim D_k$. The diffeomorphisms $j^l\overline{\eta}, j^k\overline{\eta} \in \mathrm{Diff}\,(\mathbb{C}^n,0)$ satisfy $(j^l\overline{\eta})_k = (j^k\overline{\eta})_k$ for any $l \geq k \geq 1$ (the action on $\mathfrak{m}/\mathfrak{m}^{k+1}$ depends on the power expansion of the diffeomorphism up to order k). We define $\eta_k = (j^k\overline{\eta})_k$ for $k \in \mathbb{N}$ and $\eta = (\eta_k)_{k\geq 1}$. Then η belongs to $\varprojlim D_k$ since

$$\pi_{l,k}(\eta_l) = \pi_{l,k}((j^l\overline{\eta})_l) = (j^l\overline{\eta})_k = (j^k\overline{\eta})_k = \eta_k$$

for all $l \geq k \geq 1$. The second equality is a consequence of $j^l\overline{\eta} \in \mathrm{Diff}\,(\mathbb{C}^n,0)$ and Remark 3.7. Resuming we associate $\eta \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ to $\overline{\eta} \in \overline{\mathrm{Diff}}(\mathbb{C}^n,0)$.

Let us describe the inverse process. Given $\eta \in \varprojlim_{k \in \mathbb{N}} D_k$ we want to interpret it in some way closer to our intuition of what a diffeomorphism is. Indeed if $\eta \in \mathrm{Diff}(\mathbb{C}^n, 0)$ then the image of x_j by the operator defined by η (cf. Equation (1)) is the jth coordinate $x_j \circ \eta$ of η. How to obtain the jth coordinate of an element $(A_k)_{k \geq 1}$ of $\varprojlim_{k \in \mathbb{N}} D_k$? We consider the sequence $(A_k(x_j + \mathfrak{m}^{k+1}))_{k \geq 1}$. Since it belongs to $\mathfrak{m} = \varprojlim \mathfrak{m}/\mathfrak{m}^{k+1}$, we can interpret $(A_k(x_j + \mathfrak{m}^{k+1}))_{k \geq 1}$ as an element η_j of $\mathfrak{m}$. Moreover $j^1\eta_1 + \mathfrak{m}^2, \ldots, j^1\eta_n + \mathfrak{m}^2$ is the image by A_1 of the basis $x_1 + \mathfrak{m}^2, \ldots, x_n + \mathfrak{m}^2$. Since A_1 is invertible, $(j^1\eta_1, \ldots, j^1\eta_n)$ is also an invertible linear map. We deduce that

$$\overline{\eta}(x_1, \ldots, x_n) := (\eta_1(x_1, \ldots, x_n), \ldots, \eta_n(x_1, \ldots, x_n))$$

belongs to $\overline{\mathrm{Diff}}(\mathbb{C}^n, 0)$.

Exercise 3.4. Show that the correspondences $\overline{\eta} \to \eta$ and $\eta \to \overline{\eta}$ described above are inverses of each other. Deduce that $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ and $\overline{\mathrm{Diff}}(\mathbb{C}^n, 0)$ are isomorphic groups.

3.3 The Jordan-Chevalley decomposition

Let us see that the Jordan-Chevalley decomposition is compatible with the inverse system $((D_k)_{k \in \mathbb{N}}, (\pi_{l,k})_{l \geq k \geq 1})$ and as a consequence formal diffeomorphisms possess a multiplicative Jordan decomposition.

Let $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ (or $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$). We already know that ϕ defines an element $(\phi_k)_{k \geq 1}$ of $\varprojlim D_k$. (cf. Equation (2)). Since $\phi_k \in \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ we can consider its semisimple-unipotent decomposition $\phi_k = \phi_{k,s}\phi_{k,u} = \phi_{k,u}\phi_{k,s}$. The elements of the decomposition belong to D_k by Chevalley's theorem.

Exercise 3.5. Let $l \geq k \geq 1$ and $A \in D_l$. Show that $\pi_{l,k}(A_s)$ is semisimple and $\pi_{l,k}(A_u)$ is unipotent.

Exercise 3.6. Let $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Show $\pi_{l,k}(\phi_{l,s}) = \phi_{k,s}$ and $\pi_{l,k}(\phi_{l,u}) = \phi_{k,u}$ for $l \geq k \geq 1$. Deduce that $(\phi_{k,s})_{k \geq 1}$ and $(\phi_{k,u})_{k \geq 1}$ define elements of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$.

Definition 3.8. Let $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We say that ϕ is semisimple if ϕ_k is semisimple (cf. Equation (2)) for any $k \in \mathbb{N}$.

Definition 3.9. Let $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We say that ϕ is unipotent if ϕ_k is unipotent (cf. Equation (2)) for any $k \in \mathbb{N}$. Given a subgroup G of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ we define G_u as its subset of unipotent elements. We say that G is unipotent if $G = G_u$. We denote by $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ the subset of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ consisting of unipotent formal diffeomorphisms.

Definition 3.10. We denote by ϕ_s (resp. ϕ_u) the element $(\phi_{k,s})_{k\geq 1}$ (resp. $(\phi_{k,u})_{k\geq 1}$) of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$.

We can summarize the previous discussion in the following result:

Proposition 3.11. *Let $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. There exist unique elements ϕ_s, ϕ_u of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ such that $\phi = \phi_s \circ \phi_u = \phi_u \circ \phi_s$, ϕ_s is semisimple and ϕ_u is unipotent.*

We generalized the multiplicative Jordan decomposition for diffeomorphisms. Anyway, it is difficult to check out whether a diffeomorphism is semisimple or unipotent by applying the definition since it depends on their actions on all the jet spaces. Let us characterize the decomposition in simpler terms.

Proposition 3.12. *Let $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. Then ϕ is unipotent if and only if $j^1\phi$ is unipotent.*

Proof. The matrix of $j^1\phi$ is the transposed of the matrix of ϕ_1. Thus ϕ_1 is unipotent if and only if $j^1\phi$ is unipotent.

We have to show that ϕ_k is unipotent for any $k \in \mathbb{N}$ if and only if ϕ_1 is unipotent. Let us prove the non-trivial implication. Consider the operator $\Delta : \mathfrak{m} \to \mathfrak{m}$ defined by $\Delta(f) = f \circ \phi - f$. The unipotence of ϕ_k is equivalent to the existence of some $l = l(k)$ such that $\Delta^l(\mathfrak{m}) \subset \mathfrak{m}^{k+1}$. Hence it suffices to show that given $k \in \mathbb{N}$ there exists $j_k \in \mathbb{N}$ such that $\Delta^{j_k}(\mathfrak{m}^k) \subset \mathfrak{m}^{k+1}$. The existence of j_1 is a consequence of the unipotence of ϕ_1.

We have

$$\Delta(fg) = (fg) \circ \phi - fg = (f \circ \phi - f)(g \circ \phi - g) + (f \circ \phi - f)g + f(g \circ \phi - g)$$

and then $\Delta(fg) = \Delta(f)\Delta(g) + \Delta(f)g + f\Delta(g)$. Given $j \geq 1$ we obtain

$$\Delta^j(fg) = \sum_{j\leq m+l,\ 0\leq m\leq j,\ 0\leq l\leq j} c_{jml}\Delta^m(f)\Delta^l(g) \tag{6}$$

where c_{jml} is a positive integer number independent of f and g for $j \leq m + l$, $0 \leq m \leq j$ and $0 \leq l \leq j$.

Suppose $\Delta^{j_k}(\mathfrak{m}^k) \subset \mathfrak{m}^{k+1}$ for some $k \in \mathbb{N}$. We define $j_{k+1} = j_k + j_1$. Let $f \in \mathfrak{m}^k$ and $g \in \mathfrak{m}$. Consider a non-vanishing coefficient $c_{j_{k+1}ml}$ in Equation (6). Then we have either $m \geq j_k$ or $l \geq j_1$. In the former case the term $c_{j_{k+1}ml}\Delta^m(f)\Delta^l(g)$ belongs to $\mathfrak{m}^{k+2} = \mathfrak{m}^{k+1}\mathfrak{m}$ whereas it belongs to $\mathfrak{m}^{k+2} = \mathfrak{m}^k\mathfrak{m}^2$ in the latter case. Anyway $\Delta^{j_{k+1}}(fg)$ belongs to $\mathfrak{m}^{k+2}$. Since any element of $\mathfrak{m}^{k+1}$ is of the form $f_1g_1 + \ldots + f_ag_a$ where $f_1, \ldots, f_a \in \mathfrak{m}^k$ and $g_1, \ldots, g_a \in \mathfrak{m}$, we deduce $\Delta^{j_{k+1}}(\mathfrak{m}^{k+1}) \subset \mathfrak{m}^{k+2}$. $\square$

Next we see that a diffeomorphism is semisimple if and only if it is diagonalizable.

Proposition 3.13. *Let $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then ϕ is semisimple if and only if there exists $\psi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\psi \circ \phi \circ \psi^{-1} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$.*

Proof. Let us prove the necessary condition. The formula $\eta(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ defines an element η of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$. We claim that the transformation η_k is semisimple for any $k \in \mathbb{N}$. Indeed $x_1^{i_1} \ldots x_n^{i_n} + \mathfrak{m}^{k+1}$ is an eigenvector of η_k of eigenvalue $\lambda_1^{i_1} \ldots \lambda_n^{i_n}$ for $1 \leq i_1 + \ldots + i_n \leq k$. Since the classes of the monomials define a basis of $\mathfrak{m}/\mathfrak{m}^{k+1}$, we deduce that there exists a basis of eigenvectors for η_k. Since ϕ_k is conjugated to η_k by a linear map, ϕ_k is semisimple for any $k \in \mathbb{N}$.

Let us show the sufficient condition. Since ϕ_1 is semisimple there exists a linear map ψ_1 such that $\psi_1 \circ j^1\phi \circ \psi_1^{-1} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. We denote $\eta(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$. Let us see that if there exists $\psi_k \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\psi_k \circ \phi \circ \psi_k^{-1}$ is equal to η modulo $\mathfrak{m}^{k+1}$ (or in other words $(\psi_k \circ \phi \circ \psi_k^{-1})_k = \eta_k$) then there exists $\psi_{k+1} \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\psi_{k+1} \circ \phi \circ \psi_{k+1}^{-1}$ is equal to η modulo $\mathfrak{m}^{k+2}$. Moreover we can choose ψ_{k+1} such that it is equal to ψ_k modulo $\mathfrak{m}^{k+1}$. This result implies that $(\psi_k)_{k \geq 1}$ defines an element of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\psi \circ \phi \circ \psi^{-1} = \eta$.

We replace ϕ with $\psi_k \circ \phi \circ \psi_k^{-1}$ without lack of generality. We say that $(i_1 \ldots i_n; l)$ is resonant and we denote $(i_1 \ldots i_n; l) \in R$ if $\lambda_l = \lambda_1^{i_1} \ldots \lambda_n^{i_n}$. We define

$$S = \left(\lambda_1 x_1 + \sum_{|\underline{i}|=k+1,\ (\underline{i};1) \notin R} a_{\underline{i}}^1 \underline{x}^{\underline{i}}, \ldots, \lambda_n x_n + \sum_{|\underline{i}|=k+1,\ (\underline{i};n) \notin R} a_{\underline{i}}^n \underline{x}^{\underline{i}} \right)$$

and

$$U = \left(x_1 + \sum_{|\underline{i}|=k+1,\ (\underline{i};1) \in R} \lambda_1^{-1} a_{\underline{i}}^1 \underline{x}^{\underline{i}}, \ldots, x_n + \sum_{|\underline{i}|=k+1,\ (\underline{i};n) \in R} \lambda_n^{-1} a_{\underline{i}}^n \underline{x}^{\underline{i}} \right).$$

We have $j^{k+1}\phi = j^{k+1}(S \circ U) = j^{k+1}(U \circ S)$. It is clear that U_{k+1} is unipotent by Proposition 3.12. Suppose that we prove the existence of $\alpha_{k+1} \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ that is equal to Id modulo $\mathfrak{m}^{k+1}$ and such that $\alpha_{k+1} \circ S \circ \alpha_{k+1}^{-1}$ coincides with η modulo $\mathfrak{m}^{k+2}$. Then it is clear that S_{k+1} is semisimple by the necessary condition and $S_{k+1}U_{k+1}$ is the Jordan-Chevalley decomposition of ϕ_{k+1}. Since ϕ_{k+1} is semisimple by hypothesis, we obtain $U_{k+1} \equiv Id$ and then $U \equiv Id$. In particular $\alpha_{k+1} \circ \phi \circ \alpha_{k+1}^{-1}$ coincides with η modulo $\mathfrak{m}^{k+2}$.

Let us diagonalize S modulo $\mathfrak{m}^{k+2}$. We define

$$\alpha_{k+1} = \left(x_1 + \sum_{|\underline{i}|=k+1,\ (\underline{i};1) \notin R} \frac{a_{\underline{i}}^1}{\lambda_1 - \underline{\lambda}^{\underline{i}}} \underline{x}^{\underline{i}}, \ldots, x_n + \sum_{|\underline{i}|=k+1,\ (\underline{i};n) \notin R} \frac{a_{\underline{i}}^n}{\lambda_n - \underline{\lambda}^{\underline{i}}} \underline{x}^{\underline{i}} \right).$$

The diffeomorphism $\alpha_{k+1} \circ S \circ \alpha_{k+1}^{-1}$ coincides with η modulo $\mathfrak{m}^{k+2}$. $\square$

3.4 Formal vector fields

We want to apply the theory of linear algebraic groups to subgroups of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$. We will associate Lie algebras to (yet to be defined) Zariski-closures of subgroups of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$. The algebraic closures are not necessarily contained in $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ even for subgroups of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$; we need to consider divergent formal diffeomorphisms in the Zariski-closure. As a consequence the Lie algebras of the Zariski-closure of a subgroup of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ can not be considered in general as Lie algebras of analytic vector fields. It is necessary to consider formal vector fields.

Let us denote by $\mathfrak{X}(\mathbb{C}^n, 0)$ the Lie algebra of (singular) local vector fields defined in the neighborhood of 0 in $\mathbb{C}^n$. An element X of $\mathfrak{X}(\mathbb{C}^n, 0)$ can be interpreted as a derivation of the $\mathbb{C}$-algebra $\mathcal{O}_n$ such that X preserves the maximal ideal of $\mathcal{O}_n$. Naturally the Lie algebra $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ of formal vector fields in n variables is the set of derivations X of $\hat{\mathcal{O}}_n$ such that $X(\mathfrak{m}) \subset \mathfrak{m}$. A formal vector field $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is determined by $X(x_1), X(x_2), \ldots, X(x_n)$. We obtain

$$X = X(x_1)\frac{\partial}{\partial x_1} + \ldots + X(x_n)\frac{\partial}{\partial x_n}. \tag{7}$$

Definition 3.14. We define L_k as the Lie algebra of derivations of the $\mathbb{C}$-algebra $\mathfrak{m}/\mathfrak{m}^{k+1}$.

Exercise 3.7. Show that L_k is the Lie algebra of D_k for any $k \in \mathbb{N}$.

Analogously as for formal diffeomorphisms the Lie algebra $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ can be understood as a projective limit $\varprojlim_{k\in\mathbb{N}} L_k$. Given $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ consider the element $(X_k)_{k\geq 1}$ that defines in $\varprojlim L_k$. Since L_k is the Lie algebra of D_k for any $k \in \mathbb{N}$, we obtain that $(\exp(X_k))_{k\geq 1}$ is a formal diffeomorphism φ. Equivalently given $t \in \mathbb{C}$ the expression

$$\exp(tX) = \left(\sum_{j=0}^{\infty} \frac{t^j}{j!} X^j(x_1), \ldots, \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j(x_n)\right) \tag{8}$$

defines the exponential of tX where $X^0(f) = f$ and $X^{j+1}(f) = X(X^j(f))$ for all $f \in \hat{\mathcal{O}}_n$ and $j \geq 0$. Equation (8) has to be interpreted as an equality of operators. On the one hand the image of x_k by the operator defined by $\exp(tX)$ is equal to $x_k \circ \exp(tX)$ by definition of operator induced by a (maybe formal) diffeomorphism. On the other hand it has to be $\sum_{j=0}^{\infty} (tX)^j(x_k)/j!$ by definition of the exponential of the operator tX.

Definition 3.15. We say that a formal vector field $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is nilpotent if $j^1 X$ is a linear nilpotent vector field (cf. Equation (7)). We denote by $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$ the set of nilpotent formal vector fields.

Remark 3.16. Let $X = (X_k)_{k\geq 1} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$. Then X_k is nilpotent (as an element of L_k) for any $k \in \mathbb{N}$ if and only if X_1 is nilpotent. This result is the analogue of Proposition 3.12 for formal vector fields. The proof is similar (but simpler) than for diffeomorphisms.

It is easier to deal with unipotent diffeomorphisms, instead of general ones, since the formal properties of formal unipotent diffeomorphisms and formal nilpotent vector fields are analogous.

Proposition 3.17 (cf. [6, 14]). *The image of $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$ by the exponential map is equal to $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ and $\exp : \hat{\mathfrak{X}}_N(\mathbb{C}^n, 0) \to \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ is a bijection.*

Proof. The fundamental remark behind the proof is that the exponential establishes a bijection from nilpotent matrices to unipotent matrices. The other ingredient is that L_k is the Lie algebra of D_k.

Consider $X = (X_k)_{k\geq 1} \in \hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$. Its exponential $\exp(X) = (\exp(X_k))_{k\geq 1}$ is a unipotent formal diffeomorphism since $\exp(X_k)$ is unipotent and belongs to D_k for any $k \in \mathbb{N}$.

Let $\phi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$. The map ϕ_k is unipotent for $k \geq 1$ by definition. The infinitesimal generator $\log \phi_k$ is nilpotent by construction. Moreover $\log \phi_k$ is in the Lie algebra of $\overline{\langle \phi_k \rangle}^z$ since this group is equal to $\{\exp(t \log \phi_k) : t \in \mathbb{C}\}$ by Proposition 2.4. Since $\overline{\langle \phi_k \rangle}^z \subset D_k$, $\log \phi_k$ belongs to the Lie algebra L_k of D_k. Therefore $(\log \phi_k)_{k\geq 1}$ is a nilpotent element of $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ whose exponential is equal to ϕ.

It is clear that the correspondences that we defined are inverse of each other. □

Definition 3.18. Given $\varphi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ we define its *infinitesimal generator* $\log \varphi$ as the unique element of $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$ such that $\varphi = \exp(\log \varphi)$. We define the 1-parameter group $(\varphi^t)_{t\in\mathbb{C}}$ by $\varphi^t = \exp(t \log \varphi)$.

It is known by results of Baker, Ecalle and Liverpool that generically the infinitesimal generator of a local diffeomorphism is a divergent vector field [1] [6] [10] (cf. Remark 3.31).

3.5 Construction of the algebraic closure

In this section we construct the Zariski-closure of a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ and describe its basic properties.

Definition 3.19. We consider the $\mathfrak{m}$-adic topology, also known as the Krull topology, in $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. The sets of the form

$$U_{k,\varphi} = \{\eta \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : j^k\eta = j^k\varphi\}$$

for $k \in \mathbb{N}$ and $\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ provide a fundamental system of open sets of the topology. A sequence $(\eta(j))_{j\in\mathbb{N}}$ in $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ converges to $\eta \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ if given any $k \in \mathbb{N}$ then there exists $m(k)$ such that $j^k\eta(m) = j^k\eta$ for any $m \geq m(k)$.

Definition 3.20. Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We define $G_k = \overline{\{\phi_k : \phi \in G\}}^z$.

Lemma 3.21. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then we obtain $\pi_{l,k}(G_l) = G_k$ for all $l \geq k \geq 1$.*

Proof. The map $\pi_{l,k} : D_l \to D_k$ is a surjective morphism of algebraic groups for $l \geq k$ by Lemma 3.5. Moreover the image by $\pi_{l,k}$ of the smallest algebraic group of $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{l+1})$ containing $\{\varphi_l : \varphi \in G\}$ is the smallest algebraic group of $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ that contains $\{\varphi_k : \varphi \in G\}$ by Remark 2.15. Hence we have $\pi_{l,k}(G_l) = G_k$ if $l \geq k$. $\square$

Definition 3.22. Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We define $\overline{G}^z$ (or $\overline{G}^{(0)}$) as $\varprojlim_{k\in\mathbb{N}} G_k$, more precisely $\overline{G}^z$ is the subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ defined by

$$\overline{G}^z = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : \varphi_k \in G_k \text{ for all } k \in \mathbb{N}\}.$$

We say that G is *pro-algebraic* if $G = \overline{G}^z$.

The group $\overline{G}^z$ is the (pro-)algebraic closure of G. It is a projective limit of algebraic groups.

Exercise 3.8. Show that the pro-algebraic closure of a subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is pro-algebraic.

The next results are technical lemmas that we use to characterize the pro-algebraic subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$.

Lemma 3.23. *Let H_k be an algebraic subgroup of D_k for $k \in \mathbb{N}$. Suppose $\pi_{l,k}(H_l) = H_k$ for all $l \geq k \geq 1$. Then $\varprojlim_{k\in\mathbb{N}} H_k$ is a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Moreover the natural map $\varprojlim H_j \to H_k$ is surjective for any $k \in \mathbb{N}$.*

Proof. The inverse limit $\varprojlim H_k$ is contained in $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) = \varprojlim D_k$.

An inverse system $(S_k)_{k\in\mathbb{N}}$ of non-empty sets and surjective maps indexed by the natural numbers satisfies that the natural projections $\varprojlim_{j\in\mathbb{N}} S_j \to S_k$ are surjective for any $k \in \mathbb{N}$. Since $(\pi_{l,k})_{|H_l} : H_l \to H_k$ is surjective for $l \geq k \geq 1$, the natural map $\varprojlim_{j\in\mathbb{N}} H_j \to H_k$ is surjective for any $k \in \mathbb{N}$. In particular we have $\{\varphi_k : \varphi \in \varprojlim H_j\} = H_k$ for $k \in \mathbb{N}$ and then $\overline{(\varprojlim H_k)}^{(0)} = \varprojlim H_k$. □

Remark 3.24. Let us consider an example. Consider the group

$$\widehat{\mathrm{Diff}}\,_1(\mathbb{C}^n,0) := \{\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) : j^1\phi = Id\}$$

of formal tangent to the identity diffeomorphisms. Denote $H_k = \{A \in D_k : \pi_{k,1}(A) = Id\}$. It is an algebraic subgroup of D_k for any $k \in \mathbb{N}$. Moreover we have $\pi_{l,k}(H_l) = H_k$ for all $l \geq k \geq 1$. Since $\widehat{\mathrm{Diff}}\,_1(\mathbb{C}^n,0) = \varprojlim H_j$, it is a pro-algebraic group by Lemma 3.23.

Corollary 3.25. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then the natural map $\varprojlim G_j \to G_k$ is surjective for any $k \in \mathbb{N}$.*

Proof. We have $\pi_{l,k}(G_l) = G_k$ if $l \geq k$ by Lemma 3.21. The result is a consequence of Lemma 3.23. □

We provide two characterizations of pro-algebraic groups in next proposition.

Proposition 3.26. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then the following conditions are equivalent:*

1. *G is pro-algebraic.*
2. *$\{\varphi_k : \varphi \in G\}$ is an algebraic matrix group for any $k \in \mathbb{N}$ and G is closed in the Krull topology.*
3. *G is of the form $\varprojlim_{k\in\mathbb{N}} H_k$ where H_k is an algebraic subgroup of D_k and $\pi_{l,k}(H_l)$ is contained in H_k for all $l \geq k \geq 1$.*

Proof. Let us prove (1) $\Longrightarrow$ (2). Suppose $G = \overline{G}^{(0)}$. We obtain $\{\varphi_k : \varphi \in G\} = G_k$ by Corollary 3.25. Moreover since $\overline{G}^{(0)}$ is closed in the Krull topology by construction, G is closed in the Krull topology.

Let us show (2) $\Longrightarrow$ (1). The group G_k is equal to $\{\varphi_k : \varphi \in G\}$ by hypothesis for any $k \in \mathbb{N}$. We claim $\overline{G}^{(0)} \subset G$. Indeed given $\varphi \in \overline{G}^{(0)}$ and $k \in \mathbb{N}$ there exists $\eta(k) \in G$ such that $\varphi_k = (\eta(k))_k$ for any $k \in \mathbb{N}$ since $\overline{G}^{(0)} = \varprojlim\{\varphi_k : \varphi \in G\}$. In particular

$\varphi = \lim_{k\to\infty} \eta(k)$ where the limit is considered in the Krull topology. Since G is closed in the Krull topology, we obtain $\varphi \in G$. The inclusion $\overline{G}^{(0)} \subset G$ implies $G = \overline{G}^{(0)}$ and hence G is pro-algebraic. Moreover we obtain $G = \overline{G}^{(0)} = \varprojlim G_k$ and then G is the form in item (3) by Lemma 3.21. We just proved (2) $\implies$ (3).

Finally let us prove (3) $\implies$ (1). We define $H_{l,k} = \pi_{l,k}(H_l)$ for $l \geq k \geq 1$. The group $H_{l,k}$ is algebraic since it is the image of an algebraic group by a morphism of algebraic groups. Since $\pi_{l',k} = \pi_{l,k} \circ \pi_{l',l}$ for $l' \geq l \geq k \geq 1$, the sequence $(H_{l,k})_{l\geq k}$ is decreasing for any $k \in \mathbb{N}$. The sequence stabilizes by the noetherianity of the ring of regular functions of an affine algebraic variety. We denote $K_k = \cap_{l\geq k} H_{l,k}$. Given $l \geq k \geq 1$ we consider $l' \geq l$ such that $K_l = H_{l',l}$ and $K_k = H_{l',k}$. Since $\pi_{l',k} = \pi_{l,k} \circ \pi_{l',l}$, we deduce $\pi_{l,k}(K_l) = K_k$ for all $l \geq k \geq 1$. The construction implies $\varprojlim K_k = \varprojlim H_k$. Thus $\varprojlim H_k$ is pro-algebraic by Lemma 3.23. □

Remark 3.27. Proposition 3.26 is very useful to show that certain groups are pro-algebraic. For example consider a family $\{G_j\}_{j\in J}$ of pro-algebraic subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Let us see that $\cap_{j\in J} G_j$ is pro-algebraic. We have

$$\pi_{l,k}(\cap_{j\in J}(G_j)_l) \subset \cap_{j\in J}(G_j)_k \text{ for all } l \geq k \geq 1 \text{ and } \cap_{j\in J} G_j = \varprojlim \cap_{j\in J}(G_j)_k.$$

Since the intersection of algebraic matrix groups is an algebraic group, the group $\cap_{j\in J} G_j$ is pro-algebraic by item (3) of Proposition 3.26.

Remark 3.28. Invariance properties typically define pro-algebraic groups. Item (3) of Proposition 3.26 provides an easy way of proving such property. Let us present an example. Consider $f_1, \ldots, f_p \in \hat{\mathcal{O}}_n$ and

$$G = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : f_j \circ \varphi \equiv f_j \text{ for all } 1 \leq j \leq p\}.$$

We define

$$H_k = \{A \in D_k : A(f_j + \mathfrak{m}^{k+1}) = f_j + \mathfrak{m}^{k+1} \text{ for all } 1 \leq j \leq p\}$$

for $k \in \mathbb{N}$. It is clear that H_k is an algebraic subgroup of D_k for $k \in \mathbb{N}$. Moreover we have $\pi_{l,k}(H_l) \subset H_k$ for $l \geq k \geq 1$. Since $f \circ \phi - f = 0$ is equivalent to $f \circ \phi - f \in \mathfrak{m}^k$ for any $k \in \mathbb{N}$, the group $\varprojlim H_k$ is equal to G. Moreover G is pro-algebraic by Lemma 3.23.

The power of item (3) of Proposition 3.26 is that in order to show that G is pro-algebraic we do not need to find $\{\varphi_k : \varphi \in G\}$ explicitly; in particular we could have $\{\varphi_k : \varphi \in G\} \subsetneq H_k$. Moreover, it allows us to exploit that a pro-algebraic group can be expressed in several ways as an inverse limit of algebraic groups.

Let us check out that the Jordan-Chevalley decomposition holds in the context of pro-algebraic groups.

Proposition 3.29. *Let $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ be an element of a pro-algebraic group G. Then ϕ_s, ϕ_u belong to G.*

Proof. We have

$$\phi \in G = \overline{G}^z = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) : \varphi_k \in G_k \text{ for all } k \in \mathbb{N}\}.$$

The transformations $\phi_{k,s}$ and $\phi_{k,u}$ belong to G_k for any $k \in \mathbb{N}$ by Chevalley's theorem. Thus $\phi_s = (\phi_{k,s})_{k\geq 1}$ and $\phi_u = (\phi_{k,u})_{k\geq 1}$ belong to $\varprojlim G_k$. □

Next we calculate the algebraic closure of a cyclic unipotent group.

Remark 3.30. Let us calculate $\overline{\langle \phi \rangle}^z$ for $\phi \in \widehat{\mathrm{Diff}}\,_u(\mathbb{C}^n,0)$. We denote $G = \langle \phi \rangle$. Since ϕ_k is unipotent for any $k \in \mathbb{N}$, the group $G_k = \overline{\langle \phi_k \rangle}^z$ is equal to the 1-parameter group $\{\phi_k^t : t \in \mathbb{C}\}$. Clearly we obtain

$$\{\exp(t \log \phi) : t \in \mathbb{C}\} \subset \overline{\langle \phi \rangle}^z.$$

Let us show the reverse inclusion. An element ψ of $\varprojlim G_j$ is of the form $(\exp(t_j \log \phi)_j)_{j\geq 1}$. In order to obtain $\overline{G}^z = \{\exp(t \log \phi) : t \in \mathbb{C}\}$ it suffices to show that $\{\exp(t \log \phi) : t \in \mathbb{C}\}$ is closed in the Krull topology. This is a consequence of the injectivity of the map

$$\begin{array}{rccc} \pi_k & : & \{\exp(t \log \phi) : t \in \mathbb{C}\} & \to & D_k \\ & & \exp(t \log \phi) & \mapsto & (\exp(t \log \phi))_k \end{array}$$

for some $k \in \mathbb{N}$. The map π_k is trivially injective for any $k \in \mathbb{N}$ if $\log \phi \equiv 0$. Otherwise consider $k \in \mathbb{N}$ such that $(\log \phi)_k \not\equiv 0$. The map π_k is injective since $(\exp(t \log \phi))_k = Id$ implies $t(\log \phi)_k = 0$ and then $t = 0$.

Remark 3.31. Let $\phi \in \widehat{\mathrm{Diff}}\,_u(\mathbb{C}^n,0)$. Since $j^1 \log \phi$ is nilpotent, it is equal to $\sum_{j=1}^{n-1} \delta_j x_{j+1} \frac{\partial}{\partial x_j}$ up to a linear change of coordinates where $\delta_j \in \{0,1\}$ for any $1 \leq j < n$. Let us define $\mathrm{ord}(x_j) = n - 1 + j$ for $1 \leq j \leq n$, $\mathrm{ord}(0) = \infty$ and then

$$\mathrm{ord}\left(\sum_{|\underline{i}|\geq 1} a_{\underline{i}} x^{\underline{i}}\right) = \min\left\{\sum_{j=1}^{n} i_j(n-1+j) : a_{\underline{i}} \neq 0\right\} \text{ if } \sum_{|\underline{i}|\geq 1} a_{\underline{i}} x^{\underline{i}} \neq 0.$$

The property $\mathrm{ord}(x_1) < \ldots < \mathrm{ord}(x_n) < 2\mathrm{ord}(x_1)$ implies $\mathrm{ord}(f) < \mathrm{ord}((\log \phi)(f))$ for any $f \in \mathfrak{m}$. The minimum possible order for a monomial is n whereas the maximum possible order for a monomial of degree less or equal than j is $(2n-1)j$. Since applying $\log \phi$ increases the order, we obtain $(\log \phi)^{(2n-1)j-n+1}(\mathfrak{m}) \subset \mathfrak{m}^{j+1}$. Thus

$$\exp(t \log \phi) = \left(\sum_{|\underline{i}| \geq 1} a_{\underline{i}}^1(t) x^{\underline{i}}, \ldots, \sum_{|\underline{i}| \geq 1} a_{\underline{i}}^n(t) x^{\underline{i}} \right)$$

satisfies $a_{\underline{i}}^k(t) \in \mathbb{C}[t]$ and $\deg a_{\underline{i}}^k \leq |\underline{i}|(2n-1) - n$ for every choice of $\underline{i}$ and k. The degree of $a_{\underline{i}}^k$ is bounded by a linear function of $|\underline{i}|$. This property induces a dichotomy: either $\exp(t \log \phi)$ converges only for t in a polar set (that is, a set of logarithmic capacity 0) or $\log \phi$ converges [17], cf. [22]. A polar set has vanishing Hausdorff dimension and in particular zero Lebesgue measure. Generically $\overline{\langle \phi \rangle}^z$ contains (many) divergent elements.

Remark 3.32. Let $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ be a semisimple formal diffeomorphism. There exists a formal change of coordinates $\psi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\psi \circ \phi \circ \psi^{-1} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ The group $\overline{\langle \phi \rangle}^z$ is equal to $\psi^{-1} \circ G_{\underline{\lambda}} \circ \psi$ (cf. Definition 2.7).

Exercise 3.9. Show the analogue of Proposition 2.12 for formal diffeomorphisms. More precisely, given $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ prove that all elements of $\overline{\langle \phi_s \rangle}^z$ commute with all elements of $\overline{\langle \phi_u \rangle}^z$ and that $\overline{\langle \phi \rangle}^z$ is the group generated by $\overline{\langle \phi_s \rangle}^z$ and $\overline{\langle \phi_u \rangle}^z$.

Definition 3.33. Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Since G_k is an algebraic group of matrices and in particular a Lie group, we can define the conected component $G_{k,0}$ of the identity in G_k. We also consider the set $G_{k,u}$ of unipotent elements of G_k.

Definition 3.34. Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. We define

$$\overline{G}_0^z = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : \varphi_k \in G_{k,0} \text{ for all } k \in \mathbb{N}\}$$

and

$$\overline{G}_u^z = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : \varphi_k \in G_{k,u} \text{ for all } k \in \mathbb{N}\}.$$

The group $\overline{G}_0^z$ is the natural candidate to connected component of Id of $\overline{G}^z$. Such a component is an algebraic group in the linear case; the analogue in the pro-algebraic case is the subject of next proposition. Moreover we will show that membership in $\overline{G}_0^z$ can be checked out on the linear part.

Proposition 3.35. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then $\overline{G}_0^z$ is a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\overline{G}_0^z = \{\varphi \in \overline{G}^z : \varphi_1 \in G_{1,0}\}$.*

Proof. Let $l \geq k \geq 1$. Since $G_{l,0}$ is algebraic, $\pi_{l,k}(G_{l,0})$ is algebraic by Proposition 2.14. The dimension of $G_k = \pi_{l,k}(G_l)$ is equal to the dimension of $\pi_{l,k}(G_{l,0})$. Since $G_{l,0}$ is connected, $\pi_{l,k}(G_{l,0})$ is connected and hence contained in $G_{k,0}$. On top of that the algebraic groups $\pi_{l,k}(G_{l,0})$ and $G_{k,0}$ have the same dimension and $G_{k,0}$ is connected, we obtain $\pi_{l,k}(G_{l,0}) = G_{k,0}$ (we just proved that given a morphism $\alpha : H \to H'$ of algebraic groups then $\alpha(H)_0 = \alpha(H_0)$, cf. [3, Chapter I.1, Corollary 1,.4, p. 47]). In particular the image by $\pi_{l,k}$ of a connected component of G_l is a connected component of G_k. The map $\pi_{l,k}$ induces a map between connected components of G_l and connected components of G_k that is clearly surjective since $\pi_{l,k}$ is surjective by Lemma 3.5. Let us show that such correspondence is injective. Consider a connected component C of G_l such that $\pi_{l,k}(C) = G_{k,0}$. Then there exists $A \in C$ such that $\pi_{l,k}(A) = Id$. Thus A is unipotent by Proposition 3.12 and it belongs to $G_{l,0}$ by Exercise 2.10. Obviously we obtain $C = G_{l,0}$.

The discussion above implies $\pi_{l,k}^{-1}(G_{k,0}) = G_{l,0}$ and $\pi_{l,k}(G_{l,0}) = G_{k,0}$ for all $l \geq k \geq 1$. We deduce $\overline{G}_0^z = \{\varphi \in \overline{G}^z : \varphi_1 \in G_{1,0}\}$.

Since $\overline{G}_0^z = \varprojlim_{k\in\mathbb{N}} G_{k,0}$ and $\pi_{l,k} : G_{l,0} \to G_{k,0}$ is surjective for all $l \geq k \geq 1$, the group $\overline{G}_0^z$ is pro-algebraic by Lemma 3.23. □

We prove next that $\overline{G}_u^z$ is a pro-algebraic group if G is solvable. The next lemma is the analogue of Lemma 2.30 for groups of local diffeomorphisms.

Lemma 3.36. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then*

- $\ell(\overline{G}^z) = \ell(G)$.
- $\exp(t \log \varphi) \in \overline{G}_u^z$ *for all* $\varphi \in \overline{G}_u^z$ *and* $t \in \mathbb{C}$.
- $\overline{G}^z = \overline{G}_u^z$ *if G is unipotent.*
- *Suppose G is solvable. Then $\overline{G}_u^z$ is a pro-algebraic normal subgroup of $\overline{G}^z$.*

The three first items were proved in [13].

Proof. We have

$$\ell(G) = \max_{k\in\mathbb{N}} \ell(\{\phi_k : \phi \in G\}) = \max_{k\in\mathbb{N}} \ell(G_k) = \ell(\overline{G}^z).$$

The first and third equalities are immediate. The second equality is a consequence of the first item of Lemma 2.30.

Given $\varphi \in \overline{G}_u^z$ the group $\overline{\langle \varphi \rangle}^z$ is contained in $\overline{G}^z$ and it is equal to $\{\exp(t \log \varphi) : t \in \mathbb{C}\}$ by Remark 3.30. Since $t \log \varphi$ is nilpotent for $t \in \mathbb{C}$, the elements of $\overline{\langle \varphi \rangle}^z$ are contained in $\overline{G}_u^z$ by Proposition 3.17.

Suppose G is unipotent. Since $\{\phi_k : \phi \in G\}$ is unipotent, its Zariski-closure G_k is unipotent for any $k \in \mathbb{N}$ by Lemma 2.30. Thus $\overline{G}^z = \varprojlim G_k$ is unipotent by Proposition 3.12.

Suppose G is solvable. The set $G_{k,u}$ is an algebraic normal connected subgroup of the solvable group G_k for any $k \in \mathbb{N}$ by Lemma 2.30. We have $\pi_{l,k}^{-1}(G_{k,u}) = G_{l,u}$ for all $l \geq k \geq 1$ by Proposition 3.12. Since $\pi_{l,k}(G_l) = G_k$ by Lemma 3.21, hence $\pi_{l,k}(G_{l,u}) = G_{k,u}$ for all $l \geq k \geq 1$. Therefore $\overline{G}_u^z = \varprojlim G_{k,u}$ is a pro-algebraic group by Lemma 3.23. Moreover since $G_{k,u}$ is normal in G_k for any $k \in \mathbb{N}$, the group $\overline{G}_u^z$ is normal in $\overline{G}^z$. □

Remark 3.37. Let G be a solvable subgroup of $\mathrm{Diff}(\mathbb{C}^n, 0)$. Since membership in $\overline{G}_0^z$ and $\overline{G}_u^z$ can be checked out in the first jet, these groups have finite codimension in $\overline{G}^z$. Indeed the kernels of the natural maps

$$\overline{G}^z \to G_1/G_{1,u} \quad \text{and} \quad \overline{G}^z \to G_1/G_{1,0}$$

are equal to $\overline{G}_u^z$ and $\overline{G}_0^z$ respectively by Propositions 3.12 and 3.35. In particular $\overline{G}^z/\overline{G}_0^z$ is a finite group.

Proposition 3.38. *Let $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Consider $m \in \mathbb{Z} \setminus \{0\}$ such that $\phi^m \in \overline{\langle \phi \rangle}_0^z$. Then we obtain $\overline{\langle \phi^m \rangle}^z = \overline{\langle \phi \rangle}_0^z$.*

Proof. We denote $G = \langle \phi \rangle$. We have $\phi_k^m \in G_{k,0}$ for any $k \in \mathbb{N}$ by definition of $\overline{G}_0^z$. Proposition 2.24 implies $\overline{\langle \phi_k^m \rangle}^z = G_{k,0}$ for any $k \in \mathbb{N}$. We obtain $\overline{\langle \phi^m \rangle}^z = \overline{G}_0^z = \overline{\langle \phi \rangle}_0^z$ by construction of the pro-algebraic closure and definition of $\overline{G}_0^z$. □

We keep reproducing parts of the theory of algebraic matrix groups for formal diffeomorphisms. Next we associate Lie algebras to pro-algebraic groups.

Definition 3.39. Let G be a subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We define the set

$$\mathfrak{g} = \{X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0) : X_k \in \mathfrak{g}_k \text{ for all } k \in \mathbb{N}\}$$

where $\mathfrak{g}_k$ is the Lie algebra of G_k. We say that $\mathfrak{g}$ is the Lie algebra of $\overline{G}^z$.

Suppose G is solvable, we define

$$\mathfrak{g}_N = \{X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0) : X_k \in \mathfrak{g}_{k,u} \text{ for all } k \in \mathbb{N}\}$$

where $\mathfrak{g}_{k,u}$ is the Lie algebra of $G_{k,u}$. We say that $\mathfrak{g}_N$ is the Lie algebra of $\overline{G}_u^z$.

Remark 3.40. There are several possible definitions of Lie algebra of $\overline{G}^z$. Namely we can proceed as in Definition 3.39 or we can consider $\{X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0) : \exp(tX) \in \overline{G}^z \ \forall\, t \in \mathbb{C}\}$. We show in next proposition that both choices are equivalent.

The Lie algebra of a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ shares analogous properties with the finite dimensional case.

Proposition 3.41 ([13, Proposition 2]). *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then $\mathfrak{g}$ is equal to $\{X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0) : \exp(tX) \in \overline{G}^z \ \forall\, t \in \mathbb{C}\}$ and $\overline{G}_0^z$ is generated by the set $\{\exp(X) : X \in \mathfrak{g}\}$. Moreover if G is unipotent then the map*

$$\exp : \mathfrak{g} \to \overline{G}^z$$

is a bijection and $\mathfrak{g}$ is a Lie algebra of nilpotent formal vector fields.

Proof. The first statement is a consequence of the definition of Lie algebra of an algebraic matrix group applied to G_k for $k \in \mathbb{N}$.

Given the map $\pi_{l,k} : G_l \to G_k$ for $l \geq k \geq 1$ we can consider the map $(d\pi_{l,k})_{Id} : \mathfrak{g}_l \to \mathfrak{g}_k$ given by the differential of $\pi_{l,k}$ at Id. It is the restriction to $\mathfrak{g}_l$ of the forgetful natural map $L_{k+1} \to L_k$. The map $(d\pi_{l,k})_{Id}$ satisfies $(d\pi_{l,k})_{Id}(\mathfrak{g}_l) \subset \mathfrak{g}_k$.

Let $A \in G_l$. The image of a small neighborhood U of A in G_l is a manifold whose dimension is the rank of $(d\pi_{l,k})_{Id}$ by the constant rank theorem (the rank of the maps $(d\pi_{l,k})_B$ for $B \in G_l$ is constant by the homogeneity of algebraic groups). We deduce that G_k is the union of countably closed (in the usual topology) sets contained in manifolds of dimension $\mathrm{rk}((d\pi_{l,k})_{Id})$. Since G_k is a smooth manifold of dimension $\dim(\mathfrak{g}_k)$ we deduce $\mathrm{rk}((d\pi_{l,k})_{Id}) = \dim(\mathfrak{g}_k)$. Otherwise we have $\mathrm{rk}((d\pi_{l,k})_{Id}) < \dim(\mathfrak{g}_k)$ and G_k is the union of countably nowhere-dense closed sets; this contradicts the Baire category theorem. Since $(d\pi_{l,k})_{Id}(\mathfrak{g}_l) \subset \mathfrak{g}_k$ and both complex vector spaces have the same dimension we obtain $(d\pi_{l,k})_{Id}(\mathfrak{g}_l) = \mathfrak{g}_k$. The last two paragraphs again describe a well-known fact about algebraic groups: the surjectivity of the differential map at Id of a surjective morphism of algebraic groups in characteristic 0 (cf. [3, Chapter II.7, p. 105]).

Since $\mathfrak{g} = \{X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0) : \exp(tX) \in \overline{G}^z$ for all $t \in \mathbb{C}\}$, the set $\{\exp(X) : X \in \mathfrak{g}\}$ is contained in $\overline{G}_0^z$. Let us show that $\overline{G}_0^z$ is generated by $\{\exp(X) : X \in \mathfrak{g}\}$. Let $\phi \in \overline{G}_0^z$. Then ϕ_1 belongs to $G_{1,0}$ and as a consequence ϕ_1 is of the form $\exp(Y_1) \circ \ldots \circ \exp(Y_p)$ for some $Y_1, \ldots, Y_p$ in $\mathfrak{g}_1$ by Proposition 2.22. Since all the maps $(d\pi_{l,k})_{Id} : \mathfrak{g}_l \to \mathfrak{g}_k$ are surjective for $l \geq k \geq 1$, the natural projection $\varprojlim \mathfrak{g}_k = \mathfrak{g} \to \mathfrak{g}_1$ is surjective. Thus there exists $X_j \in \mathfrak{g}$ such that it induces the derivation Y_j of $\mathfrak{m}/\mathfrak{m}^2$ for any $1 \leq j \leq p$. The diffeomorphism

$$\psi := \exp(-X_p) \circ \ldots \exp(-X_1) \circ \phi$$

has identity linear part by construction. We are done since $\log \psi$ belongs to $\mathfrak{g}$ by Remark 3.30.

Suppose G is unipotent. Hence $\overline{G}^z$ is unipotent by Lemma 3.36. The Lie algebra $\mathfrak{g}_1$ of the unipotent group G_1 consists of nilpotent matrices. Since $\varprojlim \mathfrak{g}_k = \mathfrak{g}$ we deduce that all elements of $\mathfrak{g}$ are nilpotent by Remark 3.16. The map $\exp : \mathfrak{g} \to \overline{G}^z$ is injective since $\exp : \hat{\mathfrak{X}}_N(\mathbb{C}^n, 0) \to \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ is injective (Proposition 3.17). Finally $\exp : \mathfrak{g} \to \overline{G}^z$ is is surjective since $\log \phi \in \mathfrak{g}$ for any $\phi \in \overline{G}^z$ by Remark 3.30. □

Remark 3.42. The term "connected component of the identity of $\overline{G}^z$" for $\overline{G}_0^z$ is completely justified. On the one hand $\overline{G}^z/\overline{G}_0^z$ is a finite group by Remark 3.37. On the other hand every element φ of $\overline{G}_0^z$ is of the form $\exp(X_1) \circ \ldots \circ \exp(X_k)$ where $X_1, \ldots, X_k \in \mathfrak{g}$ by Proposition 3.41. Hence $\exp(tX_1) \circ \ldots \circ \exp(tX_k)$ describes a path connecting the identity with φ in $\overline{G}_0^z$ when t varies in $[0, 1]$.

Corollary 3.43. *Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Then $\mathfrak{g}_N$ is a complex Lie algebra of nilpotent formal vector fields such that*

$$\exp : \mathfrak{g}_N \to \overline{G}_u^z$$

is a bijection.

Proof. Denote $H = \overline{G}_u^z$. Then H is a solvable unipotent pro-algebraic group by Lemma 3.36. Since $\mathfrak{g}_N$ is the Lie algebra of H, the result is a consequence of Proposition 3.41. □

3.6 Normal forms

Let us present in the next sections some simple consequences of the previous constructions. They are easily deduced from the Jordan-Chevalley decomposition and the properties of pro-algebraic groups.

Let $\phi \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$. We can obtain a weak formal normal form for ϕ by linearizing its semisimple part. Next, we use this strategy to obtain the theorem of formal diagonalization of local diffeomorphisms with almost no calculations.

Proposition 3.44. *Let $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$. Then there exists a non-semisimple $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $j^1\phi = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ if and only if there exists a multi-index $\underline{i} \in (\mathbb{N} \cup \{0\})^n$ such that $|\underline{i}| \geq 2$ and $\underline{\lambda}^{\underline{i}} = \lambda_j$ for some $1 \leq j \leq n$.*

Proof. We denote $L(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$. Let us show the necessary condition. We have that for

$$\phi := L \circ (x_1, \dots, x_{j-1}, x_j + x^{\underline{i}}, x_{j+1}, \dots, x_n)$$

the right hand side is its Jordan-Chevally decomposition since the diffeomorphisms in the right hand side commute. Hence ϕ is not semisimple.

Suppose there exists a non-semisimple $\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ with $j^1\phi = L$. By Proposition 3.13 (and its proof) there exists $\psi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $j^1\psi = Id$ and $\psi \circ \phi_s \circ \psi^{-1} = L$. We denote $\hat{\phi}_u = \psi \circ \phi_u \circ \psi^{-1}$. The formal diffeomorphism

$$\hat{\phi}_u(x_1, \dots, x_n) = \left(x_1 + \sum_{|\underline{i}| \geq 2} a_{\underline{i}}^1(t) x^{\underline{i}}, \dots, x_n + \sum_{|\underline{i}| \geq 2} a_{\underline{i}}^n(t) x^{\underline{i}} \right)$$

is non-trivial and commutes with L. Hence we obtain $\underline{\lambda}^{\underline{i}} = \lambda_j$ for any multi-index $\underline{i}$ such that $a_{\underline{i}}^j \neq 0$. □

Let us consider the case $n = 1$. Fix $\lambda \in \mathbb{C}^*$. Then any element ϕ of $\mathrm{Diff}\,(\mathbb{C}, 0)$ (or $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$) such that $j^1\phi = \lambda x$ is formally linearizable if and only if λ is not a root of the unit.

3.7 Transferring properties to infinitesimal generators

Let us show a well-known property of unipotent diffeomorphisms. The result can be easily proved without considering pro-algebraic groups, but anyway the theory provides an easy conceptual proof.

Lemma 3.45. *Let $\phi, \psi \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$. Then $[\log\phi, \log\psi] = 0$ if and only if ϕ commutes with ψ.*

Proof. The definition of Lie bracket implies that $[\log\phi, \log\psi] = 0$ if and only if

$$\exp(t \log\phi) \circ \exp(s \log\psi) = \exp(s \log\psi) \circ \exp(t \log\phi)$$

for all $t, s \in \mathbb{C}$. This implies immediately the sufficient condition. Let us show the necessary condition.

The centralizer $Z(\psi) = \{\eta \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : \psi \circ \eta \equiv \eta \circ \psi\}$ is a pro-algebraic group containing ϕ (cf. Remark 3.28). In particular it contains $\overline{\langle\phi\rangle}^z$. Thus we obtain

$$\psi \circ \exp(t \log\phi) = \exp(t \log\phi) \circ \psi$$

for any $t \in \mathbb{C}$. We deduce $\exp(t \log\phi) \circ \exp(s \log\psi) = \exp(s \log\psi) \circ \exp(t \log\phi)$ for all $t, s \in \mathbb{C}$ analogously. Therefore $[\log\phi, \log\psi]$ vanishes. □

3.8 First integrals

Let us see how the theory of pro-algebraic groups can dramatically simplify some proofs regarding invariance properties.

Proposition 3.46. *Let us consider n elements $f_1, \dots, f_n$ of the field of fractions of $\hat{\mathcal{O}}_n$. Suppose $df_1 \wedge \dots \wedge df_n \not\equiv 0$. Then the group*

$$G = \{\phi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : f_j \circ \phi \equiv f_j \text{ for all } 1 \le j \le n\}$$

is finite.

Proof. The group G is pro-algebraic. This result is proved for $f_1, \dots, f_n \in \hat{\mathcal{O}}_n$ in Remark 3.28. The general case can be showed analogously.

Consider an element $X = \sum_{j=1}^{n} a_j \partial/\partial x_j$ in the Lie algebra $L(G)$ of G. By definition we have

$$f_j \circ \exp(tX) \equiv f_j \text{ for all } t \in \mathbb{C} \implies X(f_j) = \lim_{t \to 0} \frac{f_j \circ \exp(tX) - f_j}{t} \equiv 0$$

for any $1 \le j \le n$. The property $X(f_j) = 0$ for any $1 \le j \le n$ is equivalent to

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $df_1 \wedge \dots \wedge df_n \not\equiv 0$, the $n \times n$ matrix in the previous equation has a non-vanishing determinant and then $X \equiv 0$. Hence $L(G)$ is trivial and $\overline{G}_0^z$ is the trivial group by Proposition 3.41. Since $G/\overline{G}_0^z$ is finite by Remark 3.37, G is finite. □

3.9 Finding invariant curves

Let us see that the Jordan-Chevalley decomposition can be used to find invariant curves for a local diffeomorphism or one of its iterates.

Let us consider first an example. We define

$$\phi(x, y) = (iye^{-xy}, ixe^{xy}).$$

Does ϕ have invariant curves? And what about ϕ^p where $p \in \mathbb{N}$?

Let $X = xy(x\partial/\partial x - y\partial/\partial y)$. Since

$$\phi(x,y) = (iy, ix) \circ (xe^{xy}, ye^{-xy}) = (xe^{xy}, ye^{-xy}) \circ (iy, ix),$$

we have $\phi_s(x,y) = (iy, ix)$ and $\phi_u(x,y) = (xe^{xy}, ye^{-xy}) = \exp(X)$. Consider $p \in \mathbb{N}$ and a germ of curve γ at $(0,0)$ such that $\phi^p(\gamma) = \gamma$. Then $G_\gamma = \{\psi \in \widehat{\text{Diff}}\,(\mathbb{C}^2, 0) : \psi(\gamma) = \gamma\}$ is a pro-algebraic group containing ϕ^p. Since $\phi_s^p \circ \phi_u^p$ is the Jordan-Chevalley decomposition of ϕ^p, we obtain $\phi_s^p, \phi_u^p \in G_\gamma$ by Proposition 3.29. Remark 3.30 implies

$$\overline{\langle \phi_u^p \rangle}^z = \overline{\langle \phi_u \rangle}^z = \{\exp(t \log \phi_u) : t \in \mathbb{C}\};$$

in particular $\phi_u \in G_\gamma$ and γ is an invariant curve of the formal vector field $\log \phi_u$. Since $\log \phi_u = X$ and $X(xy) \equiv 0$, we deduce that the axis are the unique curves that are invariant by $\log \phi_u$. Therefore both axis are 2-periodic and no other curve is invariant or even periodic by ϕ.

Let us show that the periods of periodic curves are uniformly bounded.

Proposition 3.47. *Let $\phi \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$. There exists $p \in \mathbb{N}$ such that $\phi^p(\gamma) = \gamma$ for any formal periodic curve γ. Moreover every formal periodic curve is invariant if $\phi \in \overline{\langle \phi \rangle}_0^z$.*

Proof. Let $p \in \mathbb{N}$ such that $\phi^p \in \overline{\langle \phi \rangle}_0^z$. Given a formal periodic curve γ consider the pro-algebraic group $G_\gamma = \{\eta \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0) : \eta(\gamma) = \gamma\}$. There exists $q \in \mathbb{N}$ such that $\phi^{pq} \in G_\gamma$. In particular we obtain $\overline{\langle \phi^{pq} \rangle}^z \subset G_\gamma$. Since $\overline{\langle \phi^{pq} \rangle}^z = \overline{\langle \phi \rangle}_0^z$ by Proposition 3.38, we deduce $\phi^p \in G_\gamma$. $\square$

4 Derived series

Solvable subgroups of $\text{Diff}\,(\mathbb{C}^n, 0)$ provide geometrical actions on a neighborhood of a point by solvable groups. A natural question is how the dimension n restricts the complexity of such actions. A simpler problem is studying wether or not the derived lengths of solvable subgroups of $\text{Diff}\,(\mathbb{C}^n, 0)$ is bounded by a function of n and if that is the case then finding the sharpest upper bound. Since a pro-algebraic group G of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ and its pro-algebraic closure $\overline{G}^{(0)}$ have the same derived length by Lemma 3.36 and the properties of $\overline{G}_0^{(0)}$ can be understood in terms of its Lie algebra, it is natural to consider this problem in the context of pro-algebraic groups.

We will see later on that the derived group of a pro-algebraic subgroup of $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is not necessarily pro-algebraic (section 5). We need to define the analogue of the derived group in the context of pro-algebraic groups.

Definition 4.1 ([13])**.** Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. By induction we define the j-closed derived group $\overline{G}^{(j)}$ of G as the closure in the Krull topology of $[\overline{G}^{(j-1)},\overline{G}^{(j-1)}]$ for any $j \in \mathbb{N}$.

Let us provide an alternate definition of the closed derived group. A pro-algebraic subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$ is a projective limit $\varprojlim_{k\in\mathbb{N}} G_k$ of algebraic groups and hence it makes sense to consider the projective limit $\varprojlim_{k\in\mathbb{N}} G_k^{(1)}$ of the derived groups. Such group is indeed the closed derived group of G.

Proposition 4.2. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then $\overline{G}^{(j)}$ is a pro-algebraic group for any $j \in \mathbb{N}\cup\{0\}$. More precisely $\{\varphi_k : \varphi \in \overline{G}^{(j)}\}$ is the algebraic matrix group $G_k^{(j)}$ for all $j \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$ and we have $\overline{G}^{(j)} = \varprojlim G_k^{(j)}$ for any $j \in \mathbb{N}\cup\{0\}$.*

Proof. The derived group of a linear algebraic group is algebraic (cf. [3, 2.3, p. 58]). As a consequence $G_k^{(j)}$ is algebraic for all $j \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$.

We define $\tilde{G}^{(j)} = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0) : \varphi_k \in G_k^{(j)} \text{ for all } k \in \mathbb{N}\}$. Since $\pi_{l,k}(G_l) = G_k$, we obtain $\pi_{l,k}(G_l^{(j)}) = G_k^{(j)}$ for all $l \geq k \geq 1$ and $j \geq 0$. The group $\tilde{G}^{(j)}$ is pro-algebraic for any $j \geq 0$ by Lemma 3.23.

The remainder of the proof is devoted to show $\overline{G}^{(j)} = \tilde{G}^{(j)}$ for any $j \geq 0$. It suffices to prove the result for $j = 1$. The inclusion $\overline{G}^{(1)} \subset \tilde{G}^{(1)}$ is clear.

Let $\varphi \in \tilde{G}^{(1)}$. Fix $k \in \mathbb{N}$. Then φ_k is a product of commutators of elements of G_k. Since $\varprojlim_{j\in\mathbb{N}} G_j \to G_k$ is surjective by Corollary 3.25, we obtain that there exists $\eta(k) \in (\overline{G}^z)^{(1)}$ such that $(\eta(k))_k = \varphi_k$. Therefore φ is the limit in the Krull topology of the sequence $(\eta(k))_{k\geq 1}$. We are done since the Krull closure of $(\overline{G}^z)^{(1)}$ is equal to $\overline{G}^{(1)}$ by definition. $\square$

The next lemma provides the analogue of the derived series for pro-algebraic groups.

Lemma 4.3. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n,0)$. Then $\overline{G}^{(j)}$ is the closure in the Krull topology of the j-derived group of $\overline{G}^{(0)}$ for any $j \in \mathbb{N}$. Moreover, the series $\ldots \triangleleft \overline{G}^{(m)} \triangleleft \ldots \triangleleft \overline{G}^{(1)} \triangleleft \overline{G}^{(0)}$ is normal.*

Proof. Since the derived series of a group is normal and $\overline{G}^{(j)} = \varprojlim G_k^{(j)}$ by Proposition 4.2, the series $\ldots \triangleleft \overline{G}^{(m)} \triangleleft \ldots \triangleleft \overline{G}^{(1)} \triangleleft \overline{G}^{(0)}$ is normal. Analogously as in Proposition 4.2 we can show that $\overline{G}^{(j)}$ is contained in the closure of $(\overline{G}^z)^{(j)}$ in the Krull topology. Since $(\overline{G}^z)^{(j)} \subset \overline{G}^{(j)}$, we deduce that $\overline{G}^{(j)}$ is the closure of $(\overline{G}^z)^{(j)}$ in the Krull topology. $\square$

Remark 4.4. The previous results justify the definition of $\overline{G}^{(j)}$. On the one hand $\overline{G}^{(j)} = \{Id\}$ is equivalent to $G^{(j)} = \{Id\}$ by Lemmas 3.36 and 4.3. On the other hand the group $\overline{G}^{(j)}$ is more compatible with the pro-algebraic nature of $\overline{G}^{z}$ than $(\overline{G}^{z})^{(j)}$ by Proposition 4.2.

Let G be a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Suppose that G is connected, i.e. $G = \overline{G}_0^{z}$. We want to obtain the Lie algebras of the closed derived groups of G. In order to complete this task we introduce some definitions for Lie algebras of formal vector fields.

Definition 4.5. The derived Lie algebra $\mathfrak{g}^{(1)}$ (or $[\mathfrak{g}, \mathfrak{g}]$) of a complex Lie algebra $\mathfrak{g}$ is the complex Lie algebra generated by the Lie brackets of elements of $\mathfrak{g}$. The derived series of $\mathfrak{g}$ is defined by setting $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(j)} := [\mathfrak{g}^{(j-1)}, \mathfrak{g}^{(j-1)}]$ for $j > 0$.

Let us introduce the closed derived series of a Lie algebra.

Definition 4.6 ([13])**.** Let $\mathfrak{g}$ be a Lie subalgebra of $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$. We denote by $\overline{\mathfrak{g}}^{(0)}$ the closure of $\mathfrak{g}$ in the Krull topology. We define the j-closed derived Lie algebra $\overline{\mathfrak{g}}^{(j)}$ of $\mathfrak{g}$ as the closure in the Krull topology of $[\overline{\mathfrak{g}}^{(j-1)}, \overline{\mathfrak{g}}^{(j-1)}]$ for any $j \in \mathbb{N}$.

In next proposition we describe the closed derives series of a Lie algebra $\mathfrak{g}$ of a pro-algebraic group G in terms of the closed derived series of G. Moreover we interprete the closed derived series of $\mathfrak{g}$ as a "projective limit" of derived series.

Proposition 4.7. *Let G be a subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\overline{G}^{z} = \overline{G}_0^{z}$. Let $\mathfrak{g}$ be the Lie algebra of $\overline{G}^{z}$. Then $\overline{\mathfrak{g}}^{(j)}$ is the Lie algebra of $\overline{G}^{(j)}$ and $\overline{G}^{(j)}$ coincides with its connected component of Id for any $j \in \mathbb{N}$. Moreover we have $\overline{\mathfrak{g}}^{(j)} = \varprojlim_{k \in \mathbb{N}} \mathfrak{g}_k^{(j)}$ for any $j \in \mathbb{N} \cup \{0\}$, where $\mathfrak{g}_k$ is the Lie algebra of G_k for any $k \in \mathbb{N}$.*

Proof. Since we work in characteristic 0 and G_k is a connected algebraic group by definition of $\overline{G}_0^{z}$, we have that $\mathfrak{g}_k^{(j)}$ is the Lie algebra of the connected algebraic group $G_k^{(j)}$ for every $j \in \mathbb{N}$ [3, Proposition 7.8, p. 108].

The Lie algebra

$$\tilde{\mathfrak{g}}^{(j)} := \{X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0) : X_k \in \mathfrak{g}_k^{(j)} \text{ for all } k \in \mathbb{N}\}$$

is the Lie algebra of $\overline{G}^{(j)}$ since

$$\overline{G}^{(j)} = \{\varphi \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0) : \varphi_k \in G_k^{(j)} \text{ for all } k \in \mathbb{N}\}$$

by Proposition 4.2. It suffices to prove $\overline{\mathfrak{g}}^{(j)} = \tilde{\mathfrak{g}}^{(j)}$ for every $j \in \mathbb{N} \cup \{0\}$. The property $\overline{\mathfrak{g}}^{(j)} \subset \tilde{\mathfrak{g}}^{(j)}$ is obvious for every $j \in \mathbb{N} \cup \{0\}$. Let us show the other inclusions.

Consider the homomorphism $(d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1} \to \mathfrak{g}_k$ defined in the proof of Proposition 3.41. Since $(d\pi_{k+1,k})_{Id}(\mathfrak{g}_{k+1}) = \mathfrak{g}_k$ for every $k \in \mathbb{N}$, the natural projection $\varprojlim \mathfrak{g}_l \to \mathfrak{g}_k$ is surjective for any $k \in \mathbb{N}$. Analogously as in the proof of Proposition 4.2 given any $X \in \tilde{\mathfrak{g}}^{(j)}$ and $k \in \mathbb{N}$ there exists $X(k) \in \mathfrak{g}^{(j)}$ such that $X(k)_k = X_k$. Hence X is the limit of the sequence $(X(k))_{k\in\mathbb{N}}$ and by definition X belongs to $\overline{\mathfrak{g}}^{(j)}$. We obtain $\tilde{\mathfrak{g}}^{(j)} \subset \overline{\mathfrak{g}}^{(j)}$ and then $\tilde{\mathfrak{g}}^{(j)} = \overline{\mathfrak{g}}^{(j)}$ for any $j \geq 0$. □

Given a connected Lie group G with Lie algebra $\mathfrak{g}$, the Lie algebra of $G^{(1)}$ is the derived Lie algebra $\mathfrak{g}^{(1)}$. Proposition 4.7 is an analogue of such result adapted to the context of connected pro-algebraic groups.

The next lemma is the analogue of Lemma 4.3 for Lie algebras.

Lemma 4.8. *Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Let $\mathfrak{g}$ be the Lie algebra of $\overline{G}^z$. Then $\overline{\mathfrak{g}}^{(j)}$ is the closure in the Krull topology of $\mathfrak{g}^{(j)}$ for any $j \in \mathbb{N}$. Moreover we have $\varphi_* \overline{\mathfrak{g}}^{(j)} = \overline{\mathfrak{g}}^{(j)}$ for all $\varphi \in \overline{G}^z$ and $j \in \mathbb{N}$. The series*

$$\ldots \triangleleft \overline{\mathfrak{g}}^{(m)} \triangleleft \ldots \triangleleft \overline{\mathfrak{g}}^{(1)} \triangleleft \overline{\mathfrak{g}}^{(0)} = \mathfrak{g}$$

is normal.

Proof. Since $\varprojlim \mathfrak{g}_l \to \mathfrak{g}_j$ is surjective for any $j \in \mathbb{N}$, $\overline{\mathfrak{g}}^{(j)}$ is contained in the closure in the Krull topology of $\mathfrak{g}^{(j)}$ for any $j \in \mathbb{N}$. Since $\mathfrak{g}^{(j)} \subset \overline{\mathfrak{g}}^{(j)}$ and $\overline{\mathfrak{g}}^{(j)}$ is closed in the Krull topology, we deduce that $\overline{\mathfrak{g}}^{(j)}$ is the closure of $\mathfrak{g}^{(j)}$ in the Krull topology for any $j \geq 0$.

The property $\varphi_* \mathfrak{g} = \mathfrak{g}$ for any $\varphi \in \overline{G}^z$ is a consequence of $\mathfrak{g}$ being the Lie algebra of $\overline{G}^z$. Since the Lie subalgebras of the derived series of $\mathfrak{g}$ are characteristic, we obtain $\varphi_* \mathfrak{g}^{(j)} = \mathfrak{g}^{(j)}$ for all $\varphi \in \overline{G}^z$ and $j \geq 0$. We get $\varphi_* \overline{\mathfrak{g}}^{(j)} = \overline{\mathfrak{g}}^{(j)}$ for all $\varphi \in \overline{G}^z$ and $j \geq 0$ by taking the Krull closures.

Since $\overline{\mathfrak{g}}^{(j)} = \varprojlim_{k\in\mathbb{N}} \mathfrak{g}_k^{(j)}$ for any $j \geq 0$ and the derived series are normal, the series

$$\ldots \triangleleft \overline{\mathfrak{g}}^{(m)} \triangleleft \ldots \triangleleft \overline{\mathfrak{g}}^{(1)} \triangleleft \overline{\mathfrak{g}}^{(0)} = \mathfrak{g}$$

is normal. □

The next proposition establishes that the derived length of a connected subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ and its Lie algebra coincide.

Proposition 4.9. *Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\overline{G}^z = \overline{G}_0^z$. Then the derived lengths of G and $\mathfrak{g}$ coincide.*

Proof. Fix $j \geq 0$. We have $G^{(j)} = \{Id\}$ if and only if $(\overline{G}^{(0)})^{(j)}$ by Lemma 3.36. Since the closure of $(\overline{G}^{(0)})^{(j)}$ in the Krull topology is equal to $\overline{G}^{(j)}$ by Lemma 4.3, we obtain $G^{(j)} = \{Id\}$ if and only if $\overline{G}^{(j)} = \{Id\}$. The Lie algebra of $\overline{G}^{(j)}$ is equal to $\overline{\mathfrak{g}}^{(j)}$ by Proposition 4.7; moreover $\exp(\overline{\mathfrak{g}}^{(j)})$ generates $\overline{G}^{(j)}$ since this group coincides with is connected component of Id (Proposition 4.7) and Proposition 3.41. Clearly $\overline{G}^{(j)} = \{Id\}$ if and only if $\overline{\mathfrak{g}}^{(j)} = 0$. Moreover $\mathfrak{g}^{(j)} = 0$ if and only if $\overline{\mathfrak{g}}^{(j)} = 0$ since $\overline{\mathfrak{g}}^{(j)} = 0$ is the closure in the Krull topology of $\mathfrak{g}^{(j)}$ by Lemma 4.8. We deduce $G^{(j)} = \{Id\}$ if and only $\mathfrak{g}^{(j)} = 0$ for any $j \geq 0$. □

This text is intended to be elementary and we will not provide the details of the calculations of sharp upper bounds of the derived length of solvable subgroups of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$. Anyway the previous ideas can be used to show the following results.

Theorem 4.10 ([13]). *Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ such that $\overline{G}_0^z = \overline{G}^z$. Then $\ell(G) \leq 2n$. Moreover there exists a subgroup H of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\overline{H}_0^z = \overline{H}^z$ and $\ell(H) = 2n$.*

Theorem 4.11 ([13]). *Let G be a unipotent solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then we have $\ell(G) \leq 2n - 1$. Moreover there exists a unipotent subgroup H of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\ell(H) = 2n - 1$.*

The next theorem is classical. As a generalization we can calculate sharpest upper bounds of the derived length of solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ for $n \leq 5$.

Theorem 4.12 (cf. [11], [9, Theorem 6.10, p. 85]). *Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$. Then $\ell(G) \leq 2$. Moreover there exists a subgroup H of $\mathrm{Diff}\,(\mathbb{C}, 0)$ such that $\ell(H) = 2$.*

Theorem 4.13 ([21]). *Fix $2 \leq n \leq 5$. Let G be a solvable subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$. Then $\ell(G) \leq 2n+1$. Moreover there exists a subgroup H of $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\ell(H) = 2n+1$.*

5 Pro-algebraic groups in dimension 1

The theory of pro-algebraic groups is powerful but so far we exhibited just a few examples of pro-algebraic groups. The situation is very simple in dimension 1 where pro-algebraic groups can be characterized. We classify all pro-algebraic subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$ in this section.

We denote by T_d the centralizer of μz in $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ where μ is a primitive d-root of the unit. An element ϕ of T_d is of the form

$$\phi(z) = \lambda z + \sum_{k=1}^{\infty} \lambda_k z^{kd+1}$$

where $\lambda \in \mathbb{C}^*$ and $\lambda_k \in \mathbb{C}$ for any $k \geq 2$.

Definition 5.1. Consider a formal vector field $X = \sum_{j=p}^{\infty} a_{j+1} z^{j+1} \frac{\partial}{\partial z} \in \hat{\mathfrak{X}}(\mathbb{C},0)$ such that $a_{p+1} \neq 0$. We define $\mathrm{ord}(X) = p$.

Remark 5.2. Given $X, Y \in \hat{\mathfrak{X}}(\mathbb{C},0)$ we have

$$\mathrm{ord}[X,Y] = \mathrm{ord}(X) + \mathrm{ord}(Y)$$

if $\mathrm{ord}(X) \neq \mathrm{ord}(Y)$.

Theorem 5.3. *Let G be a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$. Then up to a formal conjugacy G is of one of the forms:*

- $G = \{\lambda z : \lambda \in H\}$ *where H is an algebraic subgroup of $\mathbb{C}^*$.*
- $G = \left\{(\lambda^k z) \circ \exp\left(t \frac{z^{p+1}}{1+\lambda z^p} \frac{\partial}{\partial z}\right) : k \in \mathbb{Z}, t \in \mathbb{C}\right\}$ *where $p \geq 1$, $\lambda^p = 1$ and $\lambda \in \mathbb{C}$.*
- $G = \left\{(\lambda z) \circ \exp\left(t z^{p+1} \frac{\partial}{\partial z}\right) : \lambda \in H, t \in \mathbb{C}\right\}$ *where $p \geq 1$ and H is an algebraic subgroup of $\mathbb{C}^*$.*
- $G \subset T_d$ *for some $d \geq 1$, there exists $k_0 \geq 0$ such that $\{\phi_{k_0 d} : \phi \in G\}$ is algebraic and $G = \{\phi \in T_d : \phi_{k_0 d} \in G_{k_0 d}\}$.*

The three first possibilities correspond to solvable pro-algebraic groups. Notice that in the last possibility the subgroup $\{\phi \in T_d : j^{k_0 d}\phi \equiv Id\}$ is contained in G. If $d = 1$ then G contains all the elements of $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ whose order of contact with the identity is higher than k_0.

Proof. Since G is pro-algebraic, the group $j^1 G$ is algebraic and then either a finite cyclic group or equal to $\{\lambda z : \lambda \in \mathbb{C}^*\}$. Suppose that G is solvable. The classification of solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ (cf. [11], [9, section $6B_3$, p. 89]) implies that G is of one of the forms describe in the first three items.

Suppose G is non-solvable. The set G_u is a pro-algebraic group since it is the intersection of the pro-algebraic groups G and $\widehat{\mathrm{Diff}}_1(\mathbb{C},0)$ (Remark 3.27). The Lie algebra $\mathfrak{g}_N$

of G_u consists of formal vector fields with vanishing linear part and is closed in the Krull topology by Corollary 3.43. We denote $K(G) = \{\mathrm{ord}(X) : X \in \mathfrak{g}_N\}$ and $d = \gcd(K(G))$. The formal centralizer of G is a cyclic group of cardinal d by a theorem of Loray [12]. Up to a formal change of coordinates we can suppose that the centralizer of G is equal to $\langle e^{2\pi i/d}z\rangle$ and then $G \subset T_d$. The set $K(G)$ satisfies $k_1 + k_2 \in K(G)$ for all $k_1, k_2 \in K(G)$ such that $k_1 \neq k_2$ by Remark 5.2. Hence it is simple to see that $K(G)$ contains all the natural numbers of the form kd for some $k_0 \in \mathbb{N}$ and any $k \geq k_0$. Since $\mathfrak{g}_N$ is closed in the Krull topology, it contains all formal vector fields of the form $z^{k_0 d+1}\tilde{g}(z^d)\frac{\partial}{\partial z}$. Thus G contains all formal diffeomorphisms of the form $z + z^{k_0 d+1}\tilde{g}(z^d)$. The group $\{\phi_{k_0 d} : \phi \in G\}$ is algebraic by Proposition 3.26. The inclusion $G \subset \{\phi \in T_d : \phi_{k_0 d} \in G_{k_0 d}\}$ is clear. Let us show the reverse inclusion. Given an element $\phi \in T_d$ such that $\phi_{k_0 d} \in G_{k_0 d}$ there exists $\eta \in G$ such that $\eta_{k_0 d} = \phi_{k_0 d}$ since G is pro-algebraic. The formal diffeomorphism $\eta^{-1} \circ \phi$ is of the form $z + z^{k_0 d+1}\tilde{g}(z^d)$ and hence it belongs to G. Since η belongs to G, ϕ belongs to G. □

The pro-algebraic solvable subgroups of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$ have the finite determination property and their dimension is 0, 1 or 2. On the other hand up to ramification a non-solvable pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$ has finite codimension. More precisely G has finite codimension in T_d for some $d \in \mathbb{N}$.

The derived group of a pro-algebraic subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ is not necessarily pro-algebraic

We justified that the closed derived series of a pro-algebraic group is the right concept instead of the derived series in section 4. But a priori these series could be the same, making the introduction of the closed derived series redundant. We show in this section that in general the series are different.

The closed derived series and the derived series coincide for any pro-algebraic group if and only if the derived group of a pro-algebraic group is always pro-algebraic. We exhibit in this section an example of a pro-algebraic subgroup G of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^3, 0)$ whose derived group is not pro-algebraic.

Consider

$$X = x\frac{\partial}{\partial y},\ Y = y\frac{\partial}{\partial z} \text{ and } Z = x\frac{\partial}{\partial z}.$$

We have $[X, Y] = Z$, $[X, Z] = 0$ and $[Y, Z] = 0$.

Let us consider sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ of vector fields defined in a neighborhood of 0 in $\mathbb{C}^3$ such that $X_j = P_j X$, $Y_j = Q_j Y$ where $P_j, Q_j \in \mathbb{C}\{x\}$ for any $j \in \mathbb{N}$. Moreover we suppose that the multiplicity of P_j and Q_j at 0 tend to ∞ when $j \to \infty$. We also want to guarantee the independence condition

$$\sum_{1\leq j,k} \lambda_{j,k} P_j Q_k = 0 \implies \lambda_{j,k} = 0 \text{ for all } j,k \geq 0 \tag{9}$$

where the left hand side is a linear combination with complex coefficients. The expression $\sum_{1\leq j,k} \lambda_{j,k} P_j Q_k$ makes sense since $P_j Q_k$ tends to 0 in the Krull topology when $j+k \to \infty$.

Lemma 5.4. *There exists a choice of homogeneous polynomials $(P_j)_{j\geq 1}$ and $(Q_j)_{j\geq 1}$ such that $\lim_{j\to\infty} P_j = 0 = \lim_{j\to\infty} Q_j$ in the Krull topology and $\deg(P_j Q_k) \neq \deg(P_{j'} Q_{k'})$ if $(j,k) \neq (j',k')$. In particular the independence condition (9) holds.*

Proof. We define $P_1 = x$ and $Q_1 = x$. Let us define $P_j = x^{a_j}$ and $Q_j = x^{b_j}$ for certain sequences $(a_j)_{j\geq 1}$ and $(b_j)_{j\geq 1}$ of natural numbers. Suppose that we already defined $P_1, Q_1, \ldots, P_j, Q_j$ for $j \geq 1$. We define

$$a_{j+1} = \max_{1\leq k,l\leq j} \deg(P_k Q_l) \text{ and then } b_{j+1} = \max_{1\leq k\leq j+1,\ 1\leq l\leq j} \deg(P_k Q_l).$$

Notice that $(a_j)_{j\geq 1}$ and $(b_j)_{j\geq 1}$ are strictly increasing.

We claim that $\deg(P_j Q_k) \neq \deg(P_{j'} Q_{k'})$ if $(j,k) \neq (j',k')$. We define

$$c_{j,k} = (\max\{2j-1, 2k\}, \min\{2j-1, 2k\}) \text{ for } j,k \in \mathbb{N}.$$

Notice that $(j,k) \neq (j',k')$ implies $c_{j,k} \neq c_{j',k'}$. Moreover if $c_{j,k} < c_{j',k'}$ in the lexicographical order then we obtain $\deg(P_j Q_k) < \deg(P_{j'} Q_{k'})$ by our choice of $(a_j)_{j\geq 1}$ and $(b_j)_{j\geq 1}$.

The equation $\sum_{1\leq j,k} \lambda_{j,k} P_j Q_k = 0$ implies $\lambda_{j,k} = 0$ for all $j,k \geq 1$ since all monomials $P_j Q_k$ with $j,k \geq 1$ have different degrees. $\square$

Definition 5.5. We denote $\lim^k_{n\to\infty} W_n = W$ if the sequence $(W_n)_{n\geq 1}$ converges to W in the Krull topology.

Consider the sets $\mathfrak{g}, \mathfrak{h} \subset \hat{\mathfrak{X}}_N(\mathbb{C}^3, 0)$ defined by

$$\mathfrak{g} = \{X \in \hat{\mathfrak{X}}_N(\mathbb{C}^3, 0) \text{ of the form } \sum_{j=1}^{\infty} \lambda_j X_j + \sum_{k=1}^{\infty} \mu_k Y_k + \sum_{m,l\geq 1} \gamma_{m,l} [X_m, Y_l]\}$$

and

$$\mathfrak{h} = \{X \in \hat{\mathfrak{X}}_N(\mathbb{C}^3, 0) : X \text{ is of the form } \sum_{m,l \geq 1} \gamma_{m,l}[X_m, Y_l]\}$$

where λ_j, μ_k and $\gamma_{m,l}$ are complex numbers.

Lemma 5.6. *$\mathfrak{g}$ is a step-2 nilpotent complex Lie algebra. Moreover $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ contained in the center of $\mathfrak{g}$ such that $\mathfrak{g}^{(1)} \subset \mathfrak{h}$ and $\mathfrak{h}$ is the closure of $\mathfrak{g}^{(1)}$ in the Krull topology.*

Proof. Let $W_1, W_2 \in \mathfrak{g}$. We have

$$W_n = \sum_{j=1}^{\infty} \lambda_{j,n} X_j + \sum_{k=1}^{\infty} \mu_{k,n} Y_k + \sum_{m,l \geq 1} \gamma_{m,l,n}[X_m, Y_l]$$

for $n \in \{1, 2\}$. The vector field

$$[W_1, W_2] = \sum_{j,k \geq 1} (\lambda_{j,1}\mu_{k,2} - \lambda_{j,2}\mu_{k,1})[X_j, Y_k]$$

belongs to $\mathfrak{g}$. The previous formula implies $[W_3, [W_1, W_2]] = 0$ for all $W_1, W_2, W_3 \in \mathfrak{g}$, the inclusion of $\mathfrak{h}$ in the center of $\mathfrak{g}$ and $\mathfrak{g}^{(1)} \subset \mathfrak{h}$. The Lie algebra $\mathfrak{g}$ is step-2 nilpotent since $\mathfrak{g}^{(1)}$ is contained in the center of $\mathfrak{g}$.

Let us prove that $\mathfrak{h}$ is closed in the Krull topology. It suffices to show that given a sequence

$$W_n = \sum_{m,l \geq 1} \gamma_{m,l,n} P_m(x) Q_l(x) x \frac{\partial}{\partial z}$$

in $\mathfrak{g}$ such that $\lim_{n\to\infty}^{k} W_n = W$ then the W belongs to $\mathfrak{h}$. Since the degrees of the monomials $P_m(x)Q_l(x)$ are pairwise different, there exists a unique sequence $(\gamma_{m,l})_{m,l\geq 1}$ such that $\sum_{m,l\geq 1} \gamma_{m,l,n} P_m(x)Q_l(x)x$ converges to $\sum_{m,l\geq 1} \gamma_{m,l} P_m(x)Q_l(x)x$ in the Krull topology when $n \to \infty$. The vector field $W = \sum_{m,l\geq 1} \gamma_{m,l} P_m(x)Q_l(x)x\partial/\partial z$ belongs to $\mathfrak{h}$.

Notice that $[X_m, Y_l]$ belongs to $\mathfrak{g}^{(1)}$ for all $m, l \geq 1$. Given an element $\sum_{m,l\geq 1} \gamma_{m,l}[X_m, Y_l]$ of $\mathfrak{h}$ the elements $\sum_{m+l\leq k} \gamma_{m,l}[X_m, Y_l]$ belong to $\mathfrak{g}^{(1)}$ and converge to $\sum \gamma_{m,l}[X_m, Y_l]$ when $k \to \infty$. We deduce that $\mathfrak{h}$ is contained in the closure of $\mathfrak{g}^{(1)}$ in the Krull topology. Since $\mathfrak{g}^{(1)} \subset \mathfrak{h}$ and $\mathfrak{h}$ is closed, $\mathfrak{h}$ is the closure of $\mathfrak{g}^{(1)}$. □

Lemma 5.7. *The complex Lie algebra $\mathfrak{g}$ is closed in the Krull topology.*

Proof. It suffices to show that given a sequence

$$W_n = \sum_{j=1}^{\infty} \lambda_{j,n} P_j(x) x \frac{\partial}{\partial y} + \sum_{k=1}^{\infty} \mu_{k,n} Q_k(x) y \frac{\partial}{\partial z} + \sum_{m,l \geq 1} \gamma_{m,l,n} P_m(x) Q_l(x) x \frac{\partial}{\partial z}$$

in $\mathfrak{g}$ such that $\lim^k_{n\to\infty} W_n = W$ then W belongs to $\mathfrak{g}$. Since $\lim^k_{n\to\infty} W_n(y) = W(y)$ and $(\deg(P_j))_{j\geq 1}$ is strictly increasing, there exists a unique sequence $(\lambda_j)_{j\geq 1}$ such that $\lim^k_{n\to\infty} \sum_{j=1}^{\infty} \lambda_{j,n} P_j(x) = \sum_{j=1}^{\infty} \lambda_j P_j(x)$. We obtain $W(y) = \sum_{j=1}^{\infty} \lambda_j P_j(x) x$. Analogously, by noticing $\lim^k_{n\to\infty} \frac{\partial W_n(z)}{\partial y} = \frac{\partial W(z)}{\partial y}$, we deduce the existence of a unique sequence $(\mu_k)_{k\geq 1}$ such that

$$\lim^k_{n\to\infty} \sum_{k=1}^{\infty} \mu_{k,n} Q_k(x) y \frac{\partial}{\partial z} = \sum_{k=1}^{\infty} \mu_k Q_k(x) y \frac{\partial}{\partial z}.$$

The previous discussion implies that the series $\sum_{m,l\geq 1} \gamma_{m,l,n} P_m(x) Q_l(x) x$ converges in the Krull topology when $n \to \infty$. Since $\mathfrak{h}$ is closed in the Krull topology by Lemma 5.6, there exists a unique sequence $(\gamma_{m,l})_{m,l\geq 1}$ such that $\sum_{m,l\geq 1} \gamma_{m,l,n} P_m(x) Q_l(x) x$ converges to $\sum_{m,l\geq 1} \gamma_{m,l} P_m(x) Q_l(x) x$ in the Krull topology when $n \to \infty$. The vector field

$$W = \sum_{j=1}^{\infty} \lambda_j P_j(x) x \frac{\partial}{\partial y} + \sum_{k=1}^{\infty} \mu_k Q_k(x) y \frac{\partial}{\partial z} + \sum_{m,l\geq 1} \gamma_{m,l} P_m(x) Q_l(x) x \frac{\partial}{\partial z}$$

belongs to $\mathfrak{g}$. $\square$

Proposition 5.8. *The set $G := exp(\mathfrak{g})$ is a pro-algebraic unipotent subgroup of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^3, 0)$ consisting of tangent to the identity elements.*

Proof. Every element of $\mathfrak{g}$ has order of contact at least 2 with 0 and then every element of G is tangent to the identity.

Since the Lie algebra $\mathfrak{g}$ is step-2 nilpotent, it can be proved that G is a group by Baker-Campbell-Hausdorff formula. It is very easy to calculate $\exp(W)$ for $W \in \mathfrak{g}$ since $W(x) = 0$, $W^2(y) = 0$ and $W^3(z) = 0$. It can be checked out that G is given by algebraic equations in every space of jets and hence $\{\phi_k : \phi \in G\}$ is an algebraic group for any $k \in \mathbb{N}$. Moreover since $\mathfrak{g}$ is closed in the Krull topology by Lemma 5.7, G is closed in the Krull topology. As a consequence G is pro-algebraic by Proposition 3.26. $\square$

Our goal is proving that $G^{(1)}$ is not a pro-algebraic group. In order to accomplish such a task let us describe $\log(G^{(1)})$).

Proposition 5.9. *The set $\log(G^{(1)})$ is equal to the Lie algebra $\mathfrak{g}^{(1)}$. Moreover $\mathfrak{g}^{(1)}$ coincides with the set of formal vector fields of the form*

$$\log \phi = \sum_{r=1}^{s} [\sum_{j=1}^{\infty} \lambda_{j,r} X_j, \sum_{k=1}^{\infty} \mu_{k,r} Y_k] \tag{10}$$

where $s \geq 0$ and $\lambda_{j,r}, \mu_{k,r} \in \mathbb{C}$ for all $j, k \geq 1$ and $1 \leq r \leq s$.

Proof. Consider elements

$$\phi_r = \exp\left(\sum_{j=1}^{\infty} \lambda_{j,r} X_j + \sum_{k=1}^{\infty} \mu_{k,r} Y_k + \sum_{m,l \geq 1} \gamma_{m,l,r} [X_m, Y_l]\right)$$

of G for $r \in \{1,2\}$. Since $\mathfrak{g}^{(1)}$ is contained in the center of $\mathfrak{g}$ we obtain

$$\log[\phi_1, \phi_2] = [\sum_{j=1}^{\infty} \lambda_{j,1} X_j, \sum_{k=1}^{\infty} \mu_{k,2} Y_k] - [\sum_{j=1}^{\infty} \lambda_{j,2} X_j, \sum_{k=1}^{\infty} \mu_{k,1} Y_k]. \tag{11}$$

Then every commutator of elements of G is of the form (10). Since $\mathfrak{g}^{(1)}$ is contained in the center of $\mathfrak{g}$, we obtain

$$\log([\phi_1, \psi_1] \circ \ldots \circ [\phi_s, \psi_s]) = \sum_{r=1}^{s} \log([\phi_r, \psi_r]) \tag{12}$$

for all $\phi_1, \psi_1, \ldots, \phi_s, \psi_s \in G$. In particular every element of $\log(G^{(1)})$ is of the form (10).

Every element ϕ of the form (10) with $s = 1$ can be obtained by considering $\lambda_{j,2} = 0$ for all $j \geq 1$ in Equation (11). We get a general element of the form (10) by applying Equation (12). □

The next step is showing that $\mathfrak{g}^{(1)}$ is not closed in the Krull topology.

Proposition 5.10. *The element $\sum_{l=1}^{\infty} [X_l, Y_l]$ belongs to the closure of $\mathfrak{g}^{(1)}$ in the Krull topology but it does not belong to $\mathfrak{g}^{(1)}$.*

Proof. Suppose that we have

$$\sum_{l=1}^{\infty} [X_l, Y_l] = \sum_{r=1}^{s} [\sum_{j=1}^{\infty} \lambda_{j,r} X_j, \sum_{k=1}^{\infty} \mu_{k,r} Y_k]. \tag{13}$$

We denote $A_r = \sum_{j=1}^{\infty} \lambda_{j,r} X_j$ and $B_r = \sum_{k=1}^{\infty} \mu_{k,r} Y_k$.

We can suppose up to multiply A_r and B_r by complex numbers that $\lambda_{1,r} \in \{0,1\}$ for any $1 \leq r \leq s$. The independence condition (9) implies $[X_1, \sum_{\lambda_{1,r}=1} B_r] = [X_1, Y_1]$ and then $\sum_{\lambda_{1,r}=1} B_r = Y_1$. Consider $1 \leq r_0 \leq s$ such that $\lambda_{1,r_0} = 1$. By replacing B_{r_0} with $Y_1 - \sum_{\lambda_{1,r}=1,\ r \neq r_0} B_r$ in Equation (13) we obtain

$$\sum_{l=2}^{\infty} [X_l, Y_l] = \sum_{\lambda_{1,r}=1,\ r \neq r_0} [A_r - A_{r_0}, B_r] + [A_{r_0} - X_1, Y_1] + \sum_{\lambda_{1,r}=0} [A_r, B_r].$$

Hence $\sum_{l=2}^{\infty}[X_l, Y_l]$ is of the form

$$\sum_{l=2}^{\infty}[X_l, Y_l] = [C_1, Y_1] + \sum_{r=2}^{s}[C_r, D_r]$$

where $C_r = \sum_{j=2}^{\infty} \lambda'_{j,r} X_j$ and $D_r = \sum_{k=1}^{\infty} \mu'_{k,r} Y_k$ for all $1 \leq r \leq s$. We define $D_1 = Y_1$. We can suppose $\mu'_{1,r} \in \{0, 1\}$ for any $2 \leq r \leq s$. We get $\sum_{\mu'_{1,r}=1} C_r = 0$ by the independence condition. We obtain

$$\sum_{l=2}^{\infty}[X_l, Y_l] = \sum_{\mu'_{1,r}=1,\ r \geq 2} [C_r, D_r - Y_1] + \sum_{\mu'_{1,r}=0} [C_r, D_r]. \tag{14}$$

All the coefficients of X_1 in C_r vanish for $1 \leq r \leq s$. Moreover the coefficient of Y_1 in $D_r - Y_1$ is 0 if $\mu'_{1,r} = 1$ and $r \geq 2$ whereas the coefficient of Y_1 in D_r vanishes if $\mu'_{1,r} = 0$. The right hand side of Equation (14) has $s - 1$ terms whereas the right hand side of Equation (13) had s terms. By repeating this process a finite number of times we deduce that there exists $l_0 \in \mathbb{N}$ such that $\sum_{l=l_0}^{\infty}[X_l, Y_l] = 0$. This contradicts the independence condition. In particular we deduce that $\sum_{l=1}^{\infty}[X_l, Y_l]$ is not of the form (13) and hence it does not belong to $\mathfrak{g}^{(1)}$.

On the other hand it is clear that $\sum_{l=1}^{j}[X_l, Y_l]$ belongs to $\mathfrak{g}^{(1)}$ for any $j \geq 1$. Since $\sum_{l=1}^{\infty}[X_l, Y_l] = \lim_{j \to \infty}^{k} \sum_{l=1}^{j}[X_l, Y_l]$, the vector field $\sum_{l=1}^{\infty}[X_l, Y_l]$ belongs to the closure of $\mathfrak{g}^{(1)}$ in the Krull topology. $\square$

Proposition 5.11. *The group $G^{(1)}$ is not pro-algebraic.*

Proof. It suffices to show that $G^{(1)}$ is not closed in the Krull topology. We are done since $G^{(1)} = \exp(\mathfrak{g}^{(1)})$ and $\mathfrak{g}^{(1)}$ is not closed in the Krull topology by Proposition 5.10. $\square$

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