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HODGE THEORY AND ELECTROMAGNETISM

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Abstract. Let M be a compact domain in $\mathbb{R}^3$. The Hodge Decomposition Theorem yields a decomposition of the space of vector fields on M into five mutually orthogonal subspaces that encode geometric and topological features of M. This decomposition is useful in many branches of mathematics, physics, and engineering. In this paper, we study the general version of this theorem, valid for differential forms on smooth, compact, oriented manifolds with boundary, in any dimension, and deduce from it the particular five-term decomposition for compact domains in 3-space. We do this by using basic notions from multivariable calculus, linear algebra, differential forms, and algebraic topology, following the article [CDTG], by Cantarella, DeTurck and Gluck, and the book of Schwarz [S]. Furthermore, we present some applications of the notions developed in this paper to the formulation of Maxwell's equations and to the graphical representations of differential forms in $\mathbb{R}^n$.

Contents

Key words and phrases. Hodge decomposition, Hodge theory, differential forms, smooth manifolds, Maxwell equations.

1. Introduction

Electromagnetism has been a source of inspiration for new mathematics, especially since the time of Maxwell. One branch of mathematics that attests to this is algebraic topology. See [GK] for an overview of this fruitful interaction. One fundamental topic in algebraic topology, strongly linked to electromagnetism, is Hodge-de Rham theory. The references [W], [BT] and [S] supply a comprehensive account of this theory and the full length of its scope. Remarkably, within the setting of de Rham cohomology and Hodge theory, Maxwell's equations can be interpreted as the property that a 2-form ω in $\mathbb{R}^4$, and its star $*\omega$ (relative to the Lorentz metric in $\mathbb{R}^4$) are both closed, that is, $d\omega = 0$ and $d * \omega = 0$ [BT]. See also Section 5.

Let F be a vector field defined on a bounded domain M in 3-space. As mentioned in [CDTG], there are many natural and fundamental questions of interest. For instance, under what conditions is F the gradient of some function? Or the curl of another vector field? In other words, when does F admit a scalar or vector potential? Is it possible to find a nonzero vector field on M that is divergence-free, curl-free, and tangent to the boundary of M? Similarly, are there any nonzero fields that are divergence-free, curl-free, and orthogonal to the boundary of M? These questions are of both theoretical and practical interest, as a student with a good understanding of multivariable calculus may recall. To answer them, one needs to understand the relationship between F and the topology of M. Indeed, as the authors in [CDTG] show, the Hodge Decomposition Theorem provides the key because it allows to decompose F into five mutually orthogonal vector fields, defined globally on M, that encode the geometric and topological features of M. Roughly speaking, a proof of this result is obtained by using the fundamental theorems of Stokes and Gauss, as well as some key results in partial differential equations and boundary problems. For details, as well as full and precise answers to the questions posed above, the reader is invited to consult [CDTG]. See also Section 4 for the statement of the Hodge decomposition for compact domains in $\mathbb{R}^3$.

The aim of this paper is to study the Hodge Decomposition theorem in its general form, i.e. for differential forms on smooth, compact, oriented manifolds with boundary (Theorems 3.10 and 3.16), and then identifying the appropriate summands of this decomposition with the ones given in [CDTG], for the particular case of three-dimensional compact domains (See Section 4). We shall do so by using basic notions from multivariable calculus, linear algebra, differential forms, and algebraic topology, following closely [CDTG] and [S]. We carry out some of the details of the proofs presented in those references, whenever possible, and build a concrete dictionary between the Hodge decomposition for solids in $\mathbb{R}^3$ and its general version for manifolds with boundary. As a result, we provide an up-to-date proof of the Hodge decomposition, accessible to an audience familiar with the language of differential forms, at least at the elementary undergraduate level described in [C, Section 7.4]. Evidence that such an enterprise can be attained, and should be pursued, is given in the article of Baryshnikov and Ghrist [BG]. The Hodge decomposition illustrates the power of Stokes' theorem in relating the boundary behaviour of a differential form with that of its exterior derivative over the whole manifold. In this context, Stokes' theorem and Green's formula are the cornerstone of the orthogonality properties described by the Hodge decomposition.

Our second goal is to provide applications of differential forms, and the Hodge star operator, to the formulation of Maxwell's equations. This should serve as an invitation for people working in electrical and industrial engineering to expand their knowledge of differential forms, and to become familiar with the general form of Stokes' theorem, since this is crucial for applications to data managing and image processing, as explained in the recent articles [BG], [ZDWT].

In subsequent work, we aim at finding concrete examples of the Hodge decomposition on some suitable family of compact manifolds, as well as providing applications of our examples and results to physics and engineering. Since we have already worked out the general case of the Hodge decomposition, we anticipate that such examples can be build explicitly from spaces such as projective spaces, and hypersurfaces embedded in spheres or in certain Grassmannians. Moreover, building examples on compactifications of Minkowski's spacetime (e.g. on $S^1 \times S^3$ or on the space of lines in $\mathbb{C}\mathrm{P}^3$) are of particular interest.

Here is an outline of the paper. Section 2 collects the main ideas and notions on differential forms and manifolds needed for our study. Of these, the Hodge star operator, the exterior derivative and the codifferential stand out as the key notions. Furthermore, we provide a quick overview of de Rham cohomology and the geometric properties it describes. In Section 3, we study the Hodge decomposition for compact manifolds with boundary. Unlike the case of manifolds without boundary, in our present situation, we need to impose boundary conditions to be able to split the spaces of differential forms into orthogonal summands. This is done by impossing Dirichlet and Neumann conditions (Definition 3.5). The main results of this section are Theorems 3.10, 3.16, 3.23 and 3.25. In Section 4, we specify and relate the general Hodge decomposition to the one given in [CDTG] for compact domains. We identify the corresponding summands and build a bridge between both versions of the decomposition. Finally, in Section 5 we apply the main notions and techniques to the formulation of Maxwell's equations and to the graphical description of differential forms, so useful in the applications of this subject to problems in engineering.

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2. Differential forms and manifolds

2.1. **Differential forms in $\mathbb{R}^n$.** The concepts of differential forms and exterior algebra are closely related to familiar notions from vector calculus. For example, they supply a way of expressing vector fields in $\mathbb{R}^3$ as either 1-forms or 2-forms, and operators such as *div*, *grad* and *curl* as exterior derivatives. Not only this is an alternative way of expression, but also a simpler one in the sense that it may reduce notation and simplify some calculations. Another advantage of using differential forms is that the applications can be extended to higher dimensions and non-Euclidean spaces, making it attractive as a supplement to vector notation in a non-euclidean setting, allowing us to enhance our understanding of, for example, electromagnetism. In this latter setting, when interested in the flux of a vector field, instead of expressing it as a 1-form it is quite convenient to write it as a flux vector, i.e as a 2-form, because such forms can be integrated over surfaces, thus emphasizing the physical nature of the flux.

2.1.1. *Basic differential forms.*

Definition 2.1. A basic **differential k-form** (or just a basic k-form) in $\mathbb{R}^n$ $(n \geq k)$ is an expression defined as follows

$$\alpha = dx_{i_1} \wedge dx_{i_2} \wedge dx_{i_3} .... \wedge dx_{i_k},$$

where $1 \leq i_j \leq n$ for $j = 1, 2, ..., k$ and $\wedge$ is an operator called exterior product or wedge product.

Some authors write only $dx_{i_2} dx_{i_3} ....dx_{i_k}$, as it is understood that it is refering to the wedge product.

For example, consider

$$\beta = dx_3$$
$$\alpha = dx_2 \wedge dx_3 \wedge dx_4$$
$$\gamma = dx_1 \wedge dx_4 \wedge dx_6.$$

α is a basic 1-form while β and γ are both basic 3-forms. We consider β as a 1-form in $\mathbb{R}^3$, where we have chosen coordinates (x_1, x_2, x_3). Note that it is possible to define the same 1-form β in $\mathbb{R}^n$ with $n > 3$, so, how do we know whether we are working in $\mathbb{R}^3$ or in $\mathbb{R}^n$ with $n > 3$? It all depends on the choice of coordinates we want to make and the space we are working on. For example, on Minkowski space-time, we work in $\mathbb{R}^4$ with basis (x_1, x_2, x_3, x_4), therefore, defining dx_6 as a 1-form in $\mathbb{R}^4$ is as impossible as defining a 5-form on the same domain. We evidence how crucial proper definition of the domain is when working with differential forms.

In $\mathbb{R}^3$ we can define the following basic 1-forms:

$$dx_1, dx_2, dx_3,$$

which can also be written as:

$$dx, dy, dz.$$

Naturally, we can make the next identification of these basic 1-forms with the basis of $\mathbb{R}^3$, so that we have:

$$i \to dx$$
$$j \to dy$$
$$k \to dz$$

We will see later how this particular identification makes it possible to represent vector fields with differential forms, which will allow to combine this mathematical framework with the classic one of vector calculus.

A basic k-form in $\mathbb{R}^n$ needs k vectors to be evaluated and provide a number. In other words, they are scalar functions that take vectors and give back a number or variable. The appropriate way of saying this is that basic k-forms are alternating multilinear maps, but we shall get to that later on.

We shall see how basic differential forms work when they are evaluated on vectors. A basic 1-form, for example, will be defined when applied to one vector. Working on $\mathbb{R}^3$, with variables x_1, x_2 and x_3, let the vector $\overline{v}$ and the 1-form α be as follows

$$\overline{v} = (v_1, v_2, v_3),$$

$$\alpha = dx_2.$$

If α is applied to the vector, then the result is:

$$\alpha_{(\overline{v})} = dx_{2(v_1,v_2,v_3)} = v_2.$$

That is, dx_2 extracts precisely the second component of the vector $\overline{v}$.

The basic k-forms, with $k \geq 1$, behave in a similar way, except for an additional step in the process, which consists of calculating the determinant of a matrix. To give an example with a basic 2-form, we are going to use the vectors $\overline{v}$ and $\overline{u} = (u_1, u_2, u_3)$. Therefore,

$$dx_1 \wedge dx_{2(\overline{v},\overline{u})} = det \begin{bmatrix} dx_{1(\overline{v})} & dx_{1(\overline{u})} \\ dx_{2(\overline{v})} & dx_{2(\overline{u})} \end{bmatrix} = det \begin{bmatrix} v_1 & u_1 \\ v_2 & u_2 \end{bmatrix}$$

Here we give a more explicit (and equivalent) definition of a basic differential k-form and how it works, according to [C].

Definition 2.2. A basic **differential k-form** on $\mathbb{R}^n$ is an expression of the form

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $1 \leq i_j \leq n$ for $j = 1, \ldots, k$. The basic k-forms are functions that require k vectors arguments $v_1, \ldots, v_k$ and are evaluated as

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1, \ldots, v_k) = det \begin{bmatrix} dx_{i_1}(v_1) & dx_{i_1}(v_2) & \ldots & dx_{i_1}(v_k) \\ dx_{i_2}(v_1) & dx_{i_2}(v_2) & \ldots & dx_{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(v_1) & dx_{i_k}(v_2) & \ldots & dx_{i_k}(v_k) \end{bmatrix}$$

From this matrix definition, it is clear that the wedge product has the following property of *anticommutativity*:

$$dx_j \wedge dx_k = -dx_k \wedge dx_j.$$

In particular,

$$dx_k \wedge dx_k = 0.$$

These properties, obtained from Definition 2.2, can be expressed by saying that a k-form is an *alternating multilinear form* on $\mathbb{R}^n$ (i.e. it is linear on each component –this follows from its definition as a determinant– and it satisfies the anticommutativity property described above). See e.g. [L] for more details.

The space of basic k-forms in $\mathbb{R}^n$ is denoted by Ω^k. When referring to the space of all basic differential forms we use $\Omega^\star$, so that $\Omega^\star = \bigoplus_{i=0}^n \Omega^k$.

We state some examples to clarify Definition 2.2. Consider the following differential forms defined on $\mathbb{R}^4$,

$$\alpha = dx_2$$
$$\beta = dx_1 \wedge dx_3$$
$$\gamma = dx_2 \wedge dx_3 \wedge dx_4,$$

and the vectors

$$a = (1, 3, 2, 1)$$
$$b = (2, 0, 1, 5)$$
$$c = (1, 2, 3, 4),$$

then

$$\alpha(a) = dx_{2(a)} = 3$$

$$\beta(a, b) = dx_1 \wedge dx_{3\ (a,b)} = det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3$$

$$\gamma(a,b,c) = dx_2 \wedge dx_3 \wedge dx_4 \ _{(a,b,c)} = det \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 5 & 4 \end{bmatrix} = -15.$$

As mentioned before, *differential forms act on vectors.* Indeed, when dx_i is evaluated in a vector, it gives back the ith-component of it, and, for k-forms where $k > 1$, the result is the determinant of an associated matrix and can be interpreted in the case of β, up to sign, as the *area* of the parallelogram generated by the projections of vectors a and b onto the plane x_1x_3. In the case of γ, the value of this form on the vectors a, b, c is the volume, up to sign, of the parallelepiped spanned by the projections of these vectors onto the 3-dimensional space $x_2x_3x_4$ (i.e. onto a copy of $\mathbb{R}^3$ sitting inside $\mathbb{R}^4$).

2.1.2. *General differential k-forms.* As mentioned before, the notion of differential forms is related to vector calculus. For example, a differential 0-form, or 0-form for short, is just another way of referring to scalar-valued functions. A differential 1-form (or just 1-form) could be seen as a vector field, but it is necessary to have some considerations before making that identification.

In this paper, the word **smooth** shall mean *differentiable of class* C^∞. Thus a function defined on an open set $U \subset \mathbb{R}^n$ with values in $\mathbb{R}^k$ is *smooth* if its partial derivatives of all orders are defined and continuous. More generally, let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be arbitrary subsets of Euclidean spaces. A map $f : X \to Y$ is called *smooth* if for each $x \in X$ there is an open set $U \subset \mathbb{R}^n$ containing x and a smooth map $F : U \to \mathbb{R}^k$ that coincides with f throughout $U \cap X$.

Definition 2.3. A general **differential** k**-form**, or just a k-form, on $\mathbb{R}^n$, is an expression of the form:

$$\omega = \sum_{i_1,\dots,i_k}^{n} f_{i_1,\dots,i_k} dx_{i_1} \wedge dx_{i_2} \wedge dx_{i_3} \dots \wedge dx_{i_k}$$

or, for short,

$$\omega = \sum_I f_I dx_I,$$

where I stands for the multi-index $(i_1, i_2, ..., i_k)$ of degree k. The f_I are all smooth functions on $\mathbb{R}^n$, named also coefficients of ω, and dx_I is representing the basic k-form of the extended expression: $dx_{i_1} \wedge dx_{i_2} ... \wedge dx_{i_k}$. In this context, we refer to k as the degree of the form ω.

In other words, the general differential forms (or smooth differential forms) are elements of the space $\Omega^\star(\mathbb{R}^n)$ defined by taking linear combinations of basic differential forms using smooth functions as coefficients, so we write

$$\Omega^\star(\mathbb{R}^n) := \{\text{smooth functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^\star.$$

That is, $\Omega^\star(\mathbb{R}^n)$ stands for all the smooth differential forms that are defined on $\mathbb{R}^n$. If we want to refer to the space of smooth k-forms, we write:

$$\Omega^k(\mathbb{R}^n).$$

Thus, with this notation, we have

$$\Omega^\star(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \Omega^k(\mathbb{R}^n).$$

Similarly, if U is an open subset of $\mathbb{R}^n$ we can define the corresponding notion of smooth differential forms on U by replacing $\mathbb{R}^n$ with U in the definitions above. Hence, the space of all differential forms on U is

$$\Omega^\star(U) = \bigoplus_{k=0}^{n} \Omega^k(U).$$

Note that a general k-form is compound by smooth functions or *coefficients*, and basic forms that act like a basis, therefore, to be evaluated as a scalar function, a general k-form needs k vectors (required by the basic forms) and a single point (required by the coefficients). For example, let us consider the following general 2-form β on $\mathbb{R}^3$:

$$\beta = (cos(x).2yz)dx_1 \wedge dx_3 + x^2\, dx_2 \wedge dx_3,$$

and vectors

$$\vec{a} = (2,3,4)$$
$$\vec{b} = (2,2,3),$$

together with the point

$$x_0 = (1,3,2).$$

Then

$$\beta_{x_0}(\vec{a},\vec{b}) = cos(1) \cdot 2 \cdot 3 \cdot 2 \cdot det \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix} + 1 \cdot det \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = -23$$

Consequently, we can think of a differential k-form ω as varying smoothly with the point $x_0 \in \mathbb{R}^n$, and on each point, taking on k vectors based at x_0 and producing a real number out of this data. See [C].

The **exterior product** or **wedge product** of two general differential forms is defined by

$$\alpha \wedge \beta = \sum_{I,J} f_I g_J dx_I dx_J,$$

where $\alpha = \sum_I f_I dx_I$ and $\beta = \sum_J g_J dx_J$ are a k-form and an l-form, respectively. Consequently, $\alpha \wedge \beta$ is a $(k+l)$-form.

From this definition, the anticommutativity property combined with the multiplication rule gives the *graded commutativity*:

$$\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta.$$

For details, see [BT]. To simplify notation, we shall sometimes write $\alpha\beta$ as a short hand for the wedge product $\alpha \wedge \beta$. We emphasize that one should proceed with care when using this simplified notation, as the product is not commutative but graded commutative.

2.1.3. *Exterior derivative.* There is a differential operator d called **exterior derivative** or exterior differentiation. This operator involves the partial derivatives of the coefficients of the form in which it is applied. The exterior derivative of a k-form gives back a $(k+1)$-form. In the following example, the exterior differentiation is applied to a 0-form f on $\mathbb{R}^n$, which results in a 1-form df whose coefficients are the partial derivatives of f.

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n$$

In general, let $\alpha = \sum_I f_I dx_I$ be a general k-form, then $d\alpha$ is a $(k+1)$-form and it is defined by

$$d\alpha = \sum_I (df_I) dx_I,$$

where I stands for the multi-index $(i_1, i_2, ..., i_k)$ of degree k. Note that the exterior derivative d is only applied to the coefficients of the form, therefore, the first step of the operation will always be calculating the exterior differentiation of each 0-form that acts as coefficient.

To give an example, let α be a general 1-form on $\mathbb{R}^3$:

$$\alpha = Pdx + Qdy + Rdz.$$

Its exterior derivative is

$$d\alpha = (dP)dx + (dQ)dy + (dR)dz$$

$$d\alpha = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) dx$$

$$d\alpha = \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial P}{\partial z}dz \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial z}dz \wedge dy + \frac{\partial R}{\partial x}dx \wedge dz + \frac{\partial R}{\partial y}dy \wedge dz$$

$$d\alpha = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

And note that it is possible to make this identification compatible with the right-hand rule:

$$i \to dy \wedge dz$$

$$j \to dz \wedge dx$$

$$k \to dx \wedge dy$$

which leads to the classic expression of the **curl** of a vector field, that has also three components on $\mathbb{R}^3$.

$$\nabla \times (P, Q, R) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\, , \, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\, , \, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Now let's define the following 2-form

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy,$$

and its exterior derivative

$$d\omega = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) dy\wedge dz + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) dz\wedge dx + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) dx\wedge dy$$

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz.$$

In the same way a 1-form has a derivative that corresponds to the *curl* of the associated vector field, now we see that the derivative of a general 2-form corresponds to the **div** operator.

Next, we state some important properties of the exterior derivative d.

Proposition 2.4 (Leibniz rule)**.** *Let α be a k-form and let β be a p-form. Then d is an antiderivation, i.e. it satisfies the Leibniz rule:*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

In particular, if f and g are 0-forms (i.e. usual smooth functions) we get

$$d(fg) = (df)g + f(dg).$$

For a proof of the Leibniz rule we refer the reader to [Sj].

Proposition 2.5. *For the exterior derivative d the following holds:*

$$d^2 = 0$$

The meaning is that when the exterior derivative is applied twice, the result is always zero. See [Sj] for a detailed proof of the proposition. It is also common terminology to say that d is *nilpotent of degree 2* (because $d^2 = 0$).

The property stated in Proposition 2.5 is similar to the ones we know from vector calculus, namely, the cancellation obtained when applying the *curl* to the *grad* of a scalar function, and the one that results from applying *div* to the *curl* of a vector field. In both operations the result is zero:

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times f) = 0$$

Later we will explain how the correspondence between vector calculus and differential forms makes both frameworks express equivalent operations.

Note that, additionally, a useful remark is that the exterior derivative of a n-form on $\mathbb{R}^n$ is a $(n+1)$-form, and therefore, it is always 0. The demonstration is straightforward and involves the property $dx_i \wedge dx_i = 0$.

Now that we know d is nilpotent of degree 2, we can consider the following two categories of k-forms: closed and exact forms.

(1) A k-form ω is **closed** when

$$d\omega = 0.$$

(2) A k-form ω is **exact** when, for some $(k-1)$-form γ, we have

$$d\gamma = \omega,$$

which implies that $d\omega = 0$, as readily seen from the identity

$$d\omega = d(d\gamma) = 0.$$

Thus, *every exact form is also closed.*

As mentioned above, these properties allow to relate vector calculus and the language of differential forms, making it possible to express in this modern language concepts such as conservative, non-conservative vector fields and potential functions. This is a step further in our goal to express the Hodge Decomposition with differential forms, as far as our scope allows.

2.1.4. *The Hodge star operator.* We know that a general differential k-form in $\mathbb{R}^n$ is a linear combination of basic differential forms and taking as coefficients smooth functions. In linear algebra terms, the basis consists of all possible basic k-forms that can be defined on $\mathbb{R}^n$. This number is obtained by selecting k unordered objects from a pile of n objects:

$$\binom{n}{k}.$$

Taking into account that two basic forms that only differ by a sign represent one component of the basis (for instance $dx_i dx_j = -dx_j dx_i$).

From these observations it follows that a natural basis $\mathcal{B}$ for $\Omega^k(\mathbb{R}^n)$ (as a module over $\mathcal{C}^\infty(\mathbb{R}^n)$) can be given in the following way

$$\mathcal{B} = \{dx_I \mid I \text{ is an increasing multi} - \text{index of degree k}\}.$$

What we mean by an increasing multi-index of degree I is that the elements $i_1, i_2, \ldots, i_k$ of I are ordered so that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Clearly, $|\mathcal{B}| = \binom{n}{k}$.

As it is known, $\binom{n}{k}$ gives back the same number obtained by selecting $(n-k)$ objects from a pile of n objects.

$$\binom{n}{k} = \binom{n}{n-k}$$

It shows that the basis conformed by basic k-forms can be substituted with the basis conformed by basic $(n-k)$-forms, as the numbers of components is equal in both cases; in more precise terms, $\Omega^k(\mathbb{R}^n)$ and $\Omega^{n-k}(\mathbb{R}^n)$ have the same dimension (for their bases have the same number of elements). For example, in vector calculus, we can express in $\mathbb{R}^3$ a vector field with (x, y, z) basis, but also with (x_1, x_2, x_3). Similarly, in the language of differential forms, (dx, dy, dz) can be used in the same way as $(dydz, dzdx, dxdy)$, as they both are basis that have same number of components and, additionally, are compatible with the right-hand rule.

There is a way of turning k-forms into $(n-k)$-forms: the Hodge star operator. If $\alpha = \sum_I f_I dx_I$ is a general k-form, the Hodge star of it, denoted $\star\alpha$, is defined by

$$\star\alpha = \sum_I f_I(\star dx_I),$$

where

$$\star dx_I = \epsilon_I dx_{I^c}.$$

Here I stands for the increasing multi-index, and I^c for the complementary increasing multi-index. This means that $\star dx_I$ is the product of all dx_i's that do not occur in dx_I, ordered so that the multi-index I^c is increasing, times a factor ϵ_I that is ± 1, chosen in such a way that the product $dx_I(\star dx_I)$ is the volume form. The volume form on $\mathbb{R}^n$ is the n-form with ordered increasing multi-index $dx_1 dx_2 dx_3 \ldots dx_n$. This all leads to the next correspondence:

$$\epsilon_I = \begin{cases} 1 & if \quad dx_I dx_{I^c} = dx_1 dx_2 dx_3 \ldots dx_n \\ -1 & if \quad dx_I dx_{I^c} = -dx_1 dx_2 dx_3 \ldots dx_n \end{cases}$$

Geometrically, the Hodge star allows to generalize our usual notions of perpendicularity or orthogonality to the setting of differential forms. Given a k-form α, its Hodge star $\star\alpha$ can be thought of as an orthogonal direction to α, since, by the very definition $\alpha \wedge \star\alpha = \pm dx_1 dx_2 \cdots dx_n$. Furthermore, if we consider α as a k-dimensional linear subspace of $\mathbb{R}^n$, then $\star\alpha$ is its orthogonal complement (under the usual inner product), and they are intimately linked. For a formalization of this approach, and several concrete examples on the Hodge star in $\mathbb{R}^n$, see [Sj]. See also section 5.2 for an application of this point of view to the graphical representation of differential forms.

The Hodge star operator extends the application of, for example, the exterior derivative and the integration of differential forms. In the first case, for example, it is possible to differentiate a n-form on $\mathbb{R}^n$, avoiding the cancellation, doing $d\star\alpha$. In the case of integration, sometimes we may have the need to express a k-form as an $(n-k)$-form so that we can calculate an integral over a certain manifold, and otherwise it would not be possible. Furthermore, it is an operator with which we will express vector calculus operators in the language of differential forms, such as the Laplace operator, the divergence and the curl of a vector field. Most importantly, it will be crucial when working out the expressions of Maxwell's equations in this more modern mathematical framework.

Lemma 2.6. *The Hodge star satisfies*

$$\star\star\, d_{x_I} = (-1)^{k(n-k)} d_{x_I},$$

where $|I| = k$.

Proof. Simply note that $\star\star\, d_{x_I} = \star(\epsilon_{x_I} dx_{I^c}) = \epsilon_I \cdot \epsilon_{I^c} \cdot dx_I$, since $(I^c)^c = I$. Finally, if $\epsilon_I = 1$, then $dx_I dx_{I^c} = dx_1 dx_2 \cdots dx_n = (-1)^{k(n-k)} dx_{I^c} dx_I$, by counting permutations. So $\epsilon_{I^c} =$

$(-1)^{k(n-k)}$. Similarly, if $\epsilon_I = -1$, then $dx_I dx_{I^c} = -dx_1 dx_2 \cdots dx_n = (-1)(-1)^{k(n-k)} dx_{I^c} dx_I$. In either case, we have

$$\epsilon_I \epsilon_{I^c} = (-1)^{k(n-k)}$$

and the claim follows. □

2.1.5. *The codifferential.* Let U be an open subset of $\mathbb{R}^n$, and let α be a k-form ($k \leq n$) defined on U. The **codifferential**, denoted δ, is the map defined by:

$$\delta : \Omega^k(U) \to \Omega^{k-1}(U)$$

$$\delta(\alpha) \;=\; (-1)^{n(k+1)+1} \star d \star \alpha$$

Namely, it is an operator that creates a $(k-1)$-form from a k-form, so that they both are defined on the same subset and are intimately related. It can be said that it is a sort of dual operator (more appropriately, an adjoint operator) to the exterior derivative; in fact, their properties are very similar.

The codifferential is nilpotent of degree 2, just as the exterior derivative, that is:

$$\delta\,\delta\,\alpha = 0.$$

Also, we can say that a k-form α is **co-closed** if

$$\delta\,\alpha = 0.$$

And it is **co-exact** if there exists a $(k+1)$-form β that satisfies

$$\delta\beta = \alpha.$$

Naturally, if a k-form is co-exact, then it is co-closed.

To illustrate the effect of the codifferential on forms, we focus on the space $\mathbb{R}^3$. Consider the following general 1-form

$$\alpha = Pdx + Qdy + Rdz.$$

It is already known that the exterior derivative represents the *curl* of the vector field. On the other hand, the codifferential represents the *div* operator. This is the demonstration:

$$\delta\alpha = (-1)^{n(k+1)+1} \star d \star \alpha = - \star d \star \alpha$$

$$\begin{aligned}
\star\alpha &= P \star dx + Q \star dy + R \star dz \\
\star\alpha &= Pdydz + Q(-dxdz) + Rdxdy \\
d \star \alpha &= P_x\, dx(dydz) + Q_y\, dy(-dxdz) + R_z\, dz(dxdy) \\
d \star \alpha &= P_x\, dxdydz + Q_y\, dxdydz + R_z\, dxdydz \\
\star d \star \alpha &= (P_x + Q_y + R_z)
\end{aligned}$$

And finally we have

$$\delta\alpha = -(P_x + Q_y + R_z)$$

Let's see how the codifferential works on different general differential forms defined on $\mathbb{R}^3$.

$\Omega^1(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^n)$	$\Omega^2(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$	$\Omega^3(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n)$
$\alpha = Pdx + Qdy + Rdz$	$\alpha = Pdydz + Qdzdx + Rdxdy$	$\alpha = Fdxdydz$
$\delta\alpha = -div(P, Q, R)$	$\delta\alpha = curl(P, Q, R) \cdot (dx, dy, dz)$	$\delta\alpha = -\nabla F \cdot (dydz, dzdx, dxdy)$

This shows the relation the codifferential has with the operators of vector calculus. Moreover, in this setting, one also defines the **Hodge-Laplacian** operator

$$\Delta : \Omega^k(U) \to \Omega^k(U)$$

given by $\Delta = d\delta + \delta d$. A k-form α is called **harmonic** if $\Delta(\alpha) = 0$. When f is a 0-form, we recover the usual notion of harmonic function: $\Delta(f) = -(f_{xx} + f_{yy} + f_{zz}) = 0$. Note that, for 0-forms, the Hodge-Laplacian is, up to a sign, the classical Laplacian operator.

2.2. **Pulling back forms.** In vector calculus, surely we may have the need to express the same vector field with different variables (defined on a new domain, i.e. to change coordinates). In a similar way, there is a transformation that allows turning k-forms in the original domain into corresponding k-forms in another one.

Let us consider a k-form w defined on a subset $M \subseteq \mathbb{R}^n$. If we need to define an equivalent k-form on another domain $N \subseteq \mathbb{R}^n$, we use the notion of pullback. First off, we must have a smooth function $T : N \to M$, which represents a smooth transformation that goes from N to M. Suppose the original k-form w is given in terms of y_i $(i = 1, ..., m)$ variables and that the transformation T is a function that express each y_i in terms of new variables $x_1, x_2, ..., x_n$, that is,

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} T_1(x_1, x_2, ..., x_n) \\ T_2(x_1, x_2, ..., x_n) \\ \vdots \\ T_m(x_1, x_2, ..., x_n) \end{pmatrix} = T.$$

Since w is a general k-form, we have

$$w = \sum_I f_I dy_I, \;\; I = i_1, i_2, ..., i_k.$$

The pullback of w along T is the k-form $T^*(w)$ defined by

$$T^*(w) = \sum_I T^*(f_I) T^*(dy_I),$$

where each term in the definition is given by

$$T^*(f_I) = f_I \circ T,$$
$$T^*(dy_I) = T^*(dy_{i_1} dy_{i_2} ... dy_{i_k}) = dT_{i_1} dT_{i_2} ... dT_{i_k}.$$

For example, suppose we have a cylinder C parametrized with variables x, y and z by

$$C = \left\{ (x, y, z) |\; x^2 + y^2 = 1,\; 0 \leq z \leq 2 \right\},$$

and a 1-form $w = xdx + ydy + zdz$ defined on it. If we want to describe w in terms of cylindrical coordinates through the transformation T,

$$T = \begin{pmatrix} x(r, \theta, z) \\ y(r, \theta, z) \\ z(r, \theta, z) \end{pmatrix} = \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \\ z \end{pmatrix},$$

then we can use the pullback of w along T,

$$T^*(w) = T^*(xdx) + T^*(ydy) + T^*(zdz)$$
$$T^*(w) = rcos\theta \; d(rcos\theta) + rsin\theta \; d(rsin\theta) + zdz$$
$$T^*(w) = rcos\theta \; (cos\theta dr - rsin\theta d\theta) + rsin\theta \; (sin\theta dr + rcos\theta d\theta) + zdz$$
$$T^*(w) = rdr + zdz.$$

The outcome is a 1-form much simpler to express compared to the initial one.

2.3. **From local to global: Manifolds.** Loosely speaking, a manifold is a set (or space) that looks locally like Euclidean space $\mathbb{R}^n$. It is the generalization of the concepts of curves and surfaces to higher dimensions. We are mostly interested in smooth manifolds, that is, manifolds obtained by gluing open subsets of $\mathbb{R}^n$ in a smooth fashion. Our usual notions of derivatives and integration can be carried out, locally, on such objects.

Before stating the most general definition of a manifold, we start by recalling the notion of parametrized manifold (see [C]), a generalization to higher dimensions of the well-known notions of parametrized curves and surfaces from multivariable calculus. These objects will play the role of local patches or charts in a more general manifold. In what follows, a bounded connected region in $\mathbb{R}^n$ refers to a subset of $\mathbb{R}^n$ that is composed of an open, connected set, possibly together with some or all of its boundary points.

Definition 2.7. Let D be a bounded connected region in $\mathbb{R}^k$. A **parametrized k-manifold** in $\mathbb{R}^n$ is a continuous map $h : D \to \mathbb{R}^n$ that is injective except possibly along the boundary ∂D. We usually refer to the image $M = h(D)$ as the **underlying manifold** of h (or the **manifold parametrized by** h).

Similar to the case of parametrized surfaces, in a k-manifold there are k coordinate curves passing through each of its points. These are produced by keeping all the variables fixed except one in the parametrization h. To be more precise, the i-th coordinate curve is

$$u_i \mapsto h(a_1, \dots, a_{i-1}, u_i, a_{i+1}, \dots, a_k),$$

where the a_j's (with $j \neq i$) are fixed constants. If h is smooth, and $h_1, h_2, \dots, h_n$ denote the component functions of h, then the tangent vector to the i-th coordinate curve is

$$T_{u_i} = \frac{\partial h}{\partial u_i} = \left(\frac{\partial h_1}{\partial u_i}, \frac{\partial h_2}{\partial u_i}, \cdots, \frac{\partial h_n}{\partial u_i} \right).$$

Quite often, for the sake of simplicity, we shall identify a parametrized manifold $h : D \to \mathbb{R}^n$ with its image $M = h(D) \subset \mathbb{R}^n$ and refer to M as a k-parametrized manifold.

Definition 2.8. Let $h : D \to \mathbb{R}^n$ be a parametrized k-manifold. We say that $M = h(D)$ is **smooth at a point** $h(u)$ if the map h is smooth at $u = (u_1, \dots, u_k) \in D \subset \mathbb{R}^k$, and if the k tangent vectors $T_{u_1}, \dots, T_{u_k}$ are linearly independent at $h(u)$. Furthermore, the parametrized k-manifold $M = h(D)$ is called **smooth** if it is smooth at every point $h(u)$, where u is in the interior of D.

Example 2.9 (See [C])**.** Let $D = [0, 1] \times [1, 2] \times [-1, 1]$. Let $h : D \to \mathbb{R}^5$ be given by

$$h(u_1, u_2, u_3) = (u_1 + u_2, 5u_2, u_2 u_3^2, u_2 - u_3, 7u_3)$$

We claim that $M = h(D)$ is a smooth parametrized 3-manifold in $\mathbb{R}^5$. Indeed, since the component functions of h are polynomials, h is smooth (and, in particular, continuous). To check that h is injective is an easy exercise, so it is left to the reader. Finally, to verify the second condition for smoothness of $M = h(D)$, note that the tangent vectors

$$\begin{aligned} T_{u_1} &= \frac{\partial h}{\partial u_1} = (1, 0, 0, 0, 0) \\ T_{u_2} &= \frac{\partial h}{\partial u_2} = (1, 5, u_3^2, 1, 0) \\ T_{u_3} &= \frac{\partial h}{\partial u_3} = (0, 0, 2u_2 u_3, -1, 7) \end{aligned}$$

can be arranged as the rows of a matrix (the transpose of the Jacobian or derivative matrix). By row reducing this matrix, we are left with three pivots. That is, the associated matrix has maximal rank, so the row vectors T_{u_i} are linearly independent at all points in D. Hence, M is smooth everywhere.

We are now ready to introduce the main notion of a manifold, objects build from parametrized manifolds by patching them together in a compatible way.

Definition 2.10. A **smooth n-dimensional manifold** is given by the following data:

(a) a Hausdorff topological space M (a space where distinct points can be separated by disjoint open sets),
(b) a collection of open sets $U_i \subset M$, where i ranges over some index set, which cover M (that is, $M = \bigcup_i U_i$),
(c) For each i a homeomorphism (a continuous bijective map whose inverse is also continuous)
$$\varphi_i : U_i \to \tilde{U}_i,$$
where $\tilde{U}_i$ is an open set in $\mathbb{R}^n$, with the property that for all i, j the composite map $\varphi_i \circ \varphi_j^{-1}$ is smooth (i.e. of class C^∞) on its domain of definition.

The maps φ_i are referred to as **charts**, **coordinate charts** or just **local coordinates**, and the entire collection of data $(U_i, \tilde{U}_i, \varphi_i)$ is called an **atlas** of charts for M.

In this context, it is also common to call the (inverse) maps φ_i^{-1} **local parametrizations** of M. They correspond to the maps h appearing in Definition 2.8, except that now we may have many different h's, depending on the part of M we want to look at.

A few remarks are needed here. First, we define φ_j^{-1} to be the obvious homeomorphism from $\tilde{U}_j$ to U_j, so $\varphi_i \circ \varphi_j^{-1}$ is well defined as a map
$$\varphi_i \circ \varphi_j^{-1} : V_{i,j} \to V_{j,i},$$
where $V_{i,j} = \varphi_j(U_i \cap U_j)$ and $V_{j,i} = \varphi_i(U_i \cap U_j)$. Since $V_{i,j}$ and $V_{j,i}$ are open subsets in $\mathbb{R}^n$, the notion of a smooth or C^∞ map, as specified in the definition, makes sense. Observe that, interchanging i and j, it follows from the definition that $\varphi_i \circ \varphi_j^{-1}$ is a homeomorphism from $V_{i,j}$ to $V_{j,i}$ with a smooth inverse.

We shall illustrate this and some of the subtleties of the definition in the next examples.

Example 2.11 (Regular level sets)**.** Let $F(x, y, z)$ be a smooth function on $\mathbb{R}^3$. Let S be the level surface $F(x, y, z) = c$. Suppose that $\nabla F(P) \neq \vec{0}$ for all $P \in S$ (that is, c is a regular value of F). By the implicit function theorem, around any point $P \in S$, it is possible to write down one of the variables as a function of the other two variables, that is, S can be written locally, around each of its points, as the graph of a function. For instance, if $F_z(P) \neq 0$, then locally around P, the surface S can be written as the graph $z = g(x, y)$ for some smooth function g. Hence, S is locally build from patches that look like $\mathbb{R}^2$, i.e. S is a two dimensional manifold. Let U_z be the set of points of S where F_z does not vanish, and let U_x be the set of points of S where F_x does not vanish. Then if a point P lies on both U_z and U_x, then it can be written as $(x, y, g(x, y))$ on U_z and as $(h(y, z), y, z)$ on U_x. In this case, the charts are $\varphi_z : U_z \to \mathbb{R}^2$, $(x, y, g(x, y)) \mapsto (x, y)$ and $\varphi_x : U_x \to \mathbb{R}^2$, $(h(y, z), y, z) \mapsto (y, z)$. Hence, the transition function between these charts (passing information from U_x to U_z) is $\varphi_z \circ \varphi_x^{-1}$, $(x, y) \mapsto (y, g(x, y))$. This function is clearly smooth, and so is its inverse.

Example 2.12 (Projective space)**.** Let $\mathbb{R}P^n$ be (real) projective space, that is the space that parametrizes lines that pass through the origin in $\mathbb{R}^{n+1}$. Since any such line can be uniquely identified by its direction vector, points of $\mathbb{R}P^n$ correspond to equivalence classes of vectors $(x_0, \ldots, x_n)$ where two vectors $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ are considered equal if $(y_0, \ldots, y_n) = (\lambda x_0, \ldots, \lambda x_n)$ for certain non-zero real number λ. The equivalence class of a vector in this setting is represented by $[x_0 : \ldots : x_n]$. For instance, in $\mathbb{R}P^2$ we have

$[1:2:-5]=[-2:-4:10]$. We say that $\mathbb{R}P^n$ comes equipped with homogeneous coordinates $[x_0 : x_1 : x_2 : ... : x_n]$. Recall that the coordinates x_i cannot be simultaneously zero (for otherwise we would not have a line!). For each $i = 0, 1, \dots, n$, let

$$U_i = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{R}P^n \mid x_i \neq 0\}$$

Clearly,

$$U_i = \left\{ \left[\frac{x_0}{x_i} : \dots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \dots : \frac{x_n}{x_i}\right] \in \mathbb{R}P^n \right\}.$$

In other words, each U_i is isomorphic to a copy of $\mathbb{R}^n$ embedded in $\mathbb{R}P^n$. It follows that we have a natural covering of $\mathbb{R}P^n$ by Euclidean spaces or affine charts:

$$\mathbb{R}P^n = \bigcup_{i=0}^{n} U_i.$$

Hence, we can view $\mathbb{R}P^n$ as a global object built up from the union of $n+1$ (overlapping) affine patches, each corresponding to a copy of $\mathbb{R}^n$ embedded in $\mathbb{R}P^n$. Note that every point of $\mathbb{R}P^n$ lies in at least one affine patch.

For instance, let us describe the situation in the particular case of $\mathbb{R}P^2$. Let $P = [x : y : z]$ be a point in $\mathbb{R}P^2$.

- If $x \neq 0$, then $[x : y : z] = [1 : \frac{y}{x} : \frac{z}{x}]$, because both coordinates represent the same direction. Therefore, we have $(\frac{y}{x}, \frac{z}{x}) \in \mathbb{R}^2$.
- If $y \neq 0$, then $[x : y : z] = [\frac{x}{y} : 1 : \frac{z}{y}]$, since they define the same direction. Therefore, $(\frac{x}{y}\ , \frac{z}{y}) \in \mathbb{R}^2$.
- If $z \neq 0$, then $[x : y : z]$ and $[\frac{x}{z} : \frac{y}{z} : 1]$ correspond to the same direction, that is, $[x : y : z] = [\frac{x}{z} : \frac{y}{z} : 1]$. We get this time, $(\frac{x}{z}\ , \frac{y}{z}) \in \mathbb{R}^2$.

Hence, $\mathbb{R}P^2$ can be seen as the union of three copies of $\mathbb{R}^2$. Moreover, note that there are transition functions between these local charts or local coordinates. Indeed, if a point $P \in \mathbb{R}P^2$ lies on both U_x and U_y, then P has two different local representations: it can be written as $[1 : \frac{y}{x} : \frac{z}{x}]$ in U_x, whereas it can be written as $[\frac{x}{y} : 1 : \frac{z}{y}]$ in U_y. Nevertheless, these two points of view are related: to go from the representation in U_x to the one in U_y we use the transition function *multiplication by* $\frac{x}{y}$. Clearly, we multiply by $\frac{y}{x}$ to pass from U_y to U_x. The general situation for arbitrary $\mathbb{R}P^n$ is essentially the same. In other words, as Definition 2.10 encodes, we can view projective space as being locally defined in terms of affine charts and transition functions that allow to transfer information between them, whenever they overlap. In summary, $\mathbb{R}P^n$ is an n-dimensional manifold.

Definition 2.13 (Manifold with boundary)**.** An n-dimensional **manifold with boundary** is an object that locally looks like the upper half space $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. That is, a manifold M is an object defined as in Definition 2.10 where the open subsets $\widetilde{U}_\alpha$ are allowed to be subsets of either $\mathbb{R}^n$ or $\mathbb{H}^n$. In other words, the standard local model for a manifold with boundary is $\mathbb{H}^n$. The charts are asked to satisfy the same compatibility conditions of Definition 2.10. We refer to [L] for details. The boundary points of M are the points whose local representation is given by letting $x_n = 0$ on the appropriate charts. We write down ∂M for the boundary of M. It is known that ∂M is an $(n-1)$-dimensional manifold (without boundary). See [L].

Example 2.14. Let $\mathbb{D}^n$ be the n-dimensional (unit) disc defined as

$$\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}.$$

The disc $\mathbb{D}^n$ is an n-dimensional manifold with boundary. The boundary of $\mathbb{D}^n$ is the $(n-1)$-dimensional unit sphere, or simply called $(n-1)$-sphere, given by

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

In symbols, we have $\partial\mathbb{D}^n = S^{n-1}$.

2.4. **Differential forms on manifolds.** According to [Sj], a general differential k-form α on M is a collection of k-forms α_i on U_i satisfying the *transformation law*

$$\alpha_j = (\varphi_i \circ \varphi_j^{-1})^*(\alpha_i)$$

on $\varphi_j(\varphi_i^{-1}(\tilde{U}_i))$. Note that $\varphi_i \circ \varphi_j^{-1}$ is nothing but a change of coordinates, as we see in the next steps. Recall that $\varphi_i : U_i \to \tilde{U}_i$, and that $M = \bigcup U_i$. It is easy to see that

$$\varphi_i \circ \varphi_j^{-1} : \tilde{U}_j \to \tilde{U}_i$$

Therefore, the pullback along $\varphi_i \circ \varphi_j^{-1}$ allows to change the coordinates of k-forms in the opposite direction

$$(\varphi_i \circ \varphi_j^{-1})^* : \Omega(\tilde{U}_i) \to \Omega(\tilde{U}_j).$$

We will require that every form defined on M can be described in either coordinates of $\tilde{U}_i$ or coordinates of $\tilde{U}_j$ whenever they overlap.

To summarize, we impose that every k-form α_i defined on a chart φ_i can be described, on $U_i \cap U_j$, as a k-form α_j on φ_j by a simple change of coordinates, via the pullback, so that $\alpha_j = (\varphi_i \circ \varphi_j^{-1})^*(\alpha_i)$. We call α a general differential k-form on M if it is a collection of k-forms α_i that satisfies such conditions, and write $\Omega^k(M)$ to denote the collection of all k-forms on M.

Example 2.15 (Differential forms on level sets or hypersurfaces)**.** Let $S = \{F(x, y, z) = c\}$ be the level surface of Example 2.11. Since ∇F does not vanish anywhere in S, then using the fact that $dF = 0$, we get

$$F_x dx + F_y dy + F_z dz = 0.$$

Taking wedge product with dy we get

$$F_x dxdy = F_z dydz$$

equivalently,

$$\frac{dxdy}{F_z} = \frac{dydz}{F_x}.$$

Proceeding in a similar fashion with the other basic differential forms dx and dz we get

$$\frac{dxdy}{F_z} = \frac{dydz}{F_x} = \frac{dzdx}{F_y}.$$

That is, the previous equation describes the compatibility conditions of a 2-form ω on the manifold S. Thus, on each coordinate chart U_x, U_y, U_z (notation as in Example 2.11), the local representation of ω is given by one of the expressions above, depending on which chart we are.

An important and illustrative exercise is to show, using the previous procedure, that $\omega = \frac{dxdy}{z}$ extends to a global 2-form on the unit sphere $S :\ x^2 + y^2 + z^2 = 1$. In global coordinates (i.e. using the coordinates of $\mathbb{R}^3$) ω can be written as

$$\omega = \frac{1}{4\pi}(xdydz - ydxdz + zdxdy).$$

The 4π has been chosen so that $\displaystyle\int_S \omega = 1$.

Definition 2.16 (Oriented manifold)**.** Let M be a compact manifold (with or without boundary). We define an *orientation* to be choice of nowhere zero top form $\omega \in \Omega^n(M)$. The n-form ω allows to define a volume form on M.

In the previous example, the 2-form ω defines an orientation on the unit sphere, and thus an area form on it.

2.5. **The Hodge star operator for manifolds.** We redefine the Hodge star operator for manifolds using the inner product of differential forms. If M is an n-dimensional oriented manifold, the inner product of two k-forms α and β is a differentiable and nonzero real-valued function $\langle \alpha, \beta \rangle$ on M. This can be done locally, see [L]. Under this procedure, the Hodge star operator

$$\star : \Omega^k(M) \to \Omega^{n-k}(M)$$

is the unique linear map from k-forms to $(n-k)$-forms such that for all $\alpha, \beta \in \Omega^k(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle vol,$$

where $\langle \alpha, \beta \rangle$ denotes the inner product of these forms, and vol the chosen volume n-form on M. Locally, in a given chart, the Hodge star operator satisfies the properties given in section 2.1.4. For more details see [L].

Arguing as in Lemma 2.6 one easily gets the following.

Lemma 2.17. *Let M be a smooth, compact, orientable, n-dimensional manifold. If ω is a k-form in M, then*

$$\star \star \omega = (-1)^{k(n-k)}\omega.$$

In other words,

$$\omega = (-1)^{k(n-k)} \star \star \omega.$$

Consequently, the Hodge star $\star$ induces an isomorphism between $\Omega^k(M)$ and $\Omega^{n-k}(M)$. This isomorphism is called **Hodge duality**. □

2.6. **Integration of differential forms over manifolds.** Informally speaking, differentiable forms are objects that take on a point and a certain number of vectors, and produce a real number. From this point of view, a smooth manifold has exactly all the information need to evaluate a differentiable form: points and the tangent vectors of the curves around each of these points.

Recall the notation from Definitions 2.7 and 2.8.

Definition 2.18. Let $D \subset \mathbb{R}^k$ be a bounded connected region, $h : D \to \mathbb{R}^n$ be the parametrization of the n-dimensional manifold M, so that $M = h(D)$, and finally let ω be a differential k-form defined on M. We define the integral of ω over M by

$$\int_M \omega = \int \cdots \int_D \omega_{h(u_1,\ldots,u_k)}(T_{u_1}, T_{u_2}, \ldots T_{u_k})\, du_1 du_2 \ldots du_k,$$

where the integral on the right hand side is the usual k-dimensional integral over D.

Though the definition has been given only for parametrized manifolds, the general case follows quite easily by taking partitions of unity. See [L] for details.

Theorem 2.19 (Generalized Stokes' Theorem)**.** *Let M be an oriented smooth n-dimensional manifold with boundary ∂M. If α is a differential $(n-1)$-form, i.e. $\alpha \in \Omega^{n-1}(M)$, then*

$$\int_M d\alpha = \int_{\partial M} i^*(\alpha),$$

where i^* denotes the pullback along the inclusion map $i : \partial M \to M$.

We refer to [W] for a detailed proof of the theorem. The theorems of Green, Stokes and Gauss all arise from 2.19, and even the fundamental theorem of calculus (if α is a 0-form). In fact, it is a generalization of the fundamentals theorems to arbitrary dimensions.

2.7. **de Rham cohomology.** Let M be a compact n-dimensional manifold (with or without boundary). The exterior derivative allows to define on $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ the structure of a chain complex. What we mean by this is the following. We have a sequence of vector spaces linked by linear maps between them, as shown below

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \Omega^2(M) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} \Omega^n(M) \longrightarrow 0,$$

where $d^k : \Omega^k(M) \to \Omega^{k+1}(M)$ is the usual exterior derivative d (the index is mainly used to place ourselves in the appropriate position along the chain). In this context, the complex or chain $(\Omega^*(M), d)$ is called the **de Rham complex**. When clear from the context, we shall drop the upper index k when referring to d^k, to simplify notation. Because $d^2 = 0$ (that is, $d^k \circ d^{k-1} = 0$), it follows that $\mathrm{Im}(d^{k-1}) \subseteq \mathrm{Ker}(d^k)$. This is nothing but a restatement of the assertion that any exact k-form is also closed.

Definition 2.20. The **k-th de Rham cohomology group** of M is given by

$$H^k_{dR}(M) = \frac{\mathrm{Ker}(d^k)}{\mathrm{Im}(d^{k-1})}.$$

The equivalence classes in $H^k_{dR}(M)$ are denoted by $[\alpha]$, where α is a closed k-form. By definition,

$$[\alpha] = \{\alpha + d\gamma \mid \gamma \in \Omega^{k-1}(M)\}.$$

Thus, the class of α can be represented by either α or by $\beta = \alpha + d\gamma$, for some $\gamma \in \Omega^{k-1}(M)$. Note that $d\gamma$ is an exact k-form. Hence, for two closed k-forms α and β of M, the assertion $[\alpha] = [\beta]$ in $H^k_{dR}(M)$ is equivalent to saying $\beta - \alpha$ is exact. That is, a closed k-form α is zero in $H^k_{dR}(M)$ if and only if α is exact. In plain words, $H^k_{dR}(M)$ measures the obstruction for a closed k-form to be exact. This parallels a familiar situation. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2 - (0,0)$. If $\nabla \times \mathbf{F} = \vec{0}$ (i.e. if $\mathbf{F}$ is closed or locally gradient), is $\mathbf{F}$ a gradient vector field? The reader probably recalls the answer right away. The answer is no. Indeed, the vector field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right),$$

satisfies $\nabla \times \mathbf{F} = \vec{0}$. However, $\mathbf{F}$ is not a gradient vector field (i.e. $\mathbf{F}$ is not exact). To see this, simply take the circle $C: \ x^2 + y^2 = 1$, oriented in a counterclockwise fashion, and check that $\int_C \mathbf{F} = 2\pi \neq 0$. Hence, $\mathbf{F}$ represents an element of $H^1_{dR}(\mathbb{R}^2 - \{(0,0)\})$. In this seemingly simple example, we can already see that the de Rham cohomology groups encode geometric information about the space (e.g. presence of holes or non-contractible loops, etc.).

Given a (real) vector space V, recall that the *dual vector space* of V, denoted V^*, consists of all linear maps from V to $\mathbb{R}$. Namely, $V^* = \{f : V \to \mathbb{R} \mid f \text{ is linear}\}$. The elements of V^* are quite often called *linear functionals*. Now let M be a smooth manifold. The k-th **singular homology group** of M, denoted $H_k(M)$ consists of the (real) vector space generated by the isomorphism classes of k-dimensional simplices of M, modulo boundaries. We refer to [L] and [B] for details. For us, it suffices to say that for a solid M in $\mathbb{R}^3$, the homology group $H_1(M)$ is generated by the equivalence classes of oriented loops in M, where two loops are declared equivalent if their difference is the boundary of an oriented surface in M. Likewise, the homology group $H_2(M)$ is generated by the equivalence classes of closed oriented surfaces in M, where two surfaces are considered equivalent if their difference is the boundary of some oriented subregion in M. If M is a solid in $\mathbb{R}^3$, then $H_3(M) = 0$ (see e.g. [E, Section 4.4]).

The dual space $(H_k(M))^*$ is, by definition, the k-th **singular cohomology of** M, and will be denoted by $H^k(M)$. (Certainly, for the learned reader, what we take here as the definition of singular cohomology is a consequence of the universal coefficients theorem, see e.g. [B].)

If M is a compact manifold, then all the cohomology groups presented here are known to be finite dimensional (real) vector spaces. See e.g. [L] or [B] for details on this claim.

At this point, the reader may wonder, is there a relation between singular cohomology and de Rham cohomology? In fact, there is some concrete evidence to support this question. Note that any class $[\eta] \in H^k_{dR}(M, \mathbb{R})$ (de Rham cohomology) defines a linear functional $I_\eta \in H^k(M, \mathbb{R})$. Indeed, given a smooth k-simplex C of M (e.g. C is a closed smooth parametrized k-submanifold of M), we can integrate η over C to produce a number. In other words, given $\eta \in H^k_{dR}(M, \mathbb{R})$ we get a natural linear map

$$I_\eta : H_k(M) \to \mathbb{R}, \quad C \mapsto \int_C \eta.$$

De Rham's theorem asserts that for a smooth manifold M, the correspondance $\eta \mapsto I_\eta$ defines an isomorphism between $H^k_{dR}(M, d)$ and the singular cohomology group $H^k(M, R)$. We refer to [W] and [L] for a proof.

Theorem 2.21 (De Rham's isomorphism theorem)**.** *Let M be a smooth manifold. Then the map*

$$I : H^k_{dR}(M) \to H^k(M, \mathbb{R}),$$

given by $\eta \mapsto I_\eta$, is an isomorphism. □

Corollary 2.22. *Let ω be a closed k-form in M. Then ω is exact if and only if*

$$\int_C \omega = 0$$

for all k-dimensional (smooth) simplices C of M.

Proof. In view of the previous theorem, ω is exact if and only if $I_\omega = 0$. The result now follows. □

It is worth pointing out that the codifferential also satisfies $\delta^2 = 0$, hence we obtain, in a similar fashion, a dual chain complex

$$0 \longrightarrow \Omega^n(M) \xrightarrow{\delta^n} \Omega^{n-1}(M) \xrightarrow{\delta^{n-1}} \Omega^{n-2}(M) \xrightarrow{\delta^{n-2}} \cdots \xrightarrow{\delta^1} \Omega^0(M) \longrightarrow 0,$$

where $\delta^k : \Omega^k(M) \to \Omega^{k-1}(M)$ is the usual codifferential. δ. Note that, unlike the de Rham complex, the degrees on the above chain decrease as we move from left to right. Once again, when clear from the context, we shall drop the upper index, to simplify notation. Because $\delta^2 = 0$, it follows that $\mathrm{Im}(\delta^{k+1}) \subset \mathrm{Ker}(\delta^k)$; in other words, any coexact k-form is also co-closed. The cohomology of this chain is defined to be

$$H^k(M, \delta) = \frac{\mathrm{Ker}(\delta^k)}{\mathrm{Im}(\delta^{k+1})}$$

We shall see in Section 3 that the cohomology of this chain computes the **relative de Rham cohomology of** X, that is,

$$H^k(M, \delta) = H^k_{dR}(M, \partial M).$$

See Theorem 3.18. In particular, we shall see that the Hodge star induces a natural isomorphism between $H^k_{dR}(M)$ and $H^{n-k}_{dR}(M, \partial M)$, also known as Poincaré duality (Theorem 3.18).

2.8. **Examples of homology and cohomology.**

(1) Given that S^2 is a Riemannian sphere,

$$S = S^2.$$

We define S^2 to be

$$S^2 = U \cup V,$$

where

$$U = S^2 - N$$
$$V = S^2 - S.$$

N and S refer to the north and south poles of the sphere, respectively. Next, we define $\alpha \in \Omega^1(S^2)$ so that

$$d\alpha = 0,$$

it is, α is closed. As $U \simeq \mathbb{R}^2$ and $V \simeq \mathbb{R}^2$, we have that α_U (restriction of α to U) is exact, since $H^1(\mathbb{R}^2) = 0$.

$$\alpha_U = df_U, \exists f_U \in \Omega^0(U).$$

Similarly,

$$\alpha_V = df_V, \exists f_V \in \Omega^0(V).$$

α is globally exact? Note that $f_U - f_V$ is defined in $U \cap V$ (connected set). Therefore,

$$d(f_U - f_V) = 0$$

As a consequence, $f_U - f_V$ is constant, so we write $f_U - f_V = c$ in $U \cap V$.

Therefore, there exists the smooth real-valued function $f : S^2 \to \mathbb{R}$, so that

$$\begin{cases} f &= f_U \text{ in } U \\ f &= f_V \text{ in } V \end{cases}$$

In other words, if $d\alpha = 0$, then $\alpha = df$ for $f \in C^0(S^2)$.

$$\therefore H^1_{dR}(S^2) = 0$$

(2) Let $S = T^2$ be a torus ($T^2 = S^1 \times S^1$). We choose angular coordinates (θ, ϕ), with $\theta, \phi \in [0, 2\pi]$.

We define the period function as

$$P : \alpha \mapsto \left(\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha \right).$$

Where P defines a lineal isomorphism between

$$P : H^1(T) \to \mathbb{R}^2,$$

because

$$\int_{\gamma_i} df = 0, \ \forall \ f$$

P is a surjective function since $(1, 0)$ and $(0, 1)$ are in the image of P. Indeed, we have

$$P : \ d\theta/2\pi \to (1, 0)$$
$$P : \ d\phi/2\pi \to (0, 1)$$

Hence, $H^1_{dR}(T) = \mathbb{Z}^2$.

In general, if S_g is a compact, oriented, surface of genus g then

$$H^k_{dR}(S_g) = \begin{cases} \mathbb{R} & \text{if} \quad k = 0, \\ \mathbb{R}^{2g} & \text{if} \quad k = 1, \\ \mathbb{R} & \text{if} \quad k = 2. \end{cases}$$

These computations and results will be useful in Section 4, since the surfaces S_g of genus g appear as boundaries of compact domains in $\mathbb{R}^3$.

3. Hodge theory

Curl and divergence are only local invariants of a vector field. They appear as summands in the linear approximation of a vector field around a point. As such, they only provide information of the local behaviour of a vector field. To understand its global behaviour, one needs to understand the summands appearing in its Hodge decomposition and/or understand the geometry of the manifold where the field is defined. This can be done, for instance, by computing the work or flux of the vector field on appropriate curves or surfaces embedded there.

3.1. Preliminary notions.

Definition 3.1 (Normal and tangential components of a differential form)**.** Let M be a smooth n-dimensional manifold with boundary ∂M. According to [GK], given a point $p \in \partial M$, one can find local coordinates $(u_1, ..., u_{n-2}, u_{n-1}, u_n)$ on M so that ∂M can be described locally as $u_n = 0$ around p. In other words, in terms of the local coordinates of M, making $u_n = 0$ gives back a local set of coordinates for ∂M. In this chart, a general k-form ω can be written as:

$$\omega = \sum_{1 \leq i_1 < i_2 < .. < i_k \leq n} f_{i_1 i_2 ... i_k} du_{i_1} du_{i_2} ... du_{i_k}.$$

On ∂M, the **tangential component of** ω is given by

$$\mathbf{t}\omega = \sum_{1 \leq i_1 < i_2 < .. < i_k < n} f_{i_1 i_2 ... i_k} du_{i_1} du_{i_2} ... du_{i_k}.$$

The only difference between the two definitions is that, in the second one, the last inequality is replaced by a strict inequality, meaning that every component with du_n is not included in $\mathbf{t}\omega$ (or making $du_n = 0$) and that every coefficient is evaluated with $u_n = 0$. Once $\mathbf{t}\omega$ is defined, **the normal component of** ω is given by

$$\mathbf{n}\omega = \omega - \mathbf{t}\omega.$$

Note that, this time, each term in $\mathbf{n}\omega$ involves dx_n, and, if not, it is zero. To conclude, ω can be written as a sum of the tangential component $\mathbf{t}\omega$ and the normal component $\mathbf{n}\omega$

$$\omega = \mathbf{t}\omega + \mathbf{n}\omega.$$

According to Schwarz, $\mathbf{t}(\omega)$ is uniquely determined by the pullback $i^*(\omega)$. So we can identify it and have

$$i^*(\omega) = i^*(\mathbf{t}(\omega)) = \mathbf{t}(\omega).$$

See [S, Proposition 1.2.6 and identity (2.26)]. If we choose Duff's definition given above (see Gross-Kotiuga), then it is immediate that $\mathbf{t}(\omega) = i^*(\omega)$. Indeed, from this point of view $\omega = \mathbf{t}(\omega) + \mathbf{n}(\omega)$ and the latter only has expressions involving dx_n, and since the boundary is given locally by $x_n = 0$, then pulling back $\mathbf{n}(\omega)$ to ∂M gives 0.

Proposition 3.2. *Let M be a smooth n-dimensional with boundary ∂M. Let $i : \partial M \to M$ be the inclusion map. Then the following identities hold:*

(1) $\mathbf{n}\omega = \omega - \mathbf{t}\omega$,
(2) $i^*(\omega) = i^*(\mathbf{t}\omega) = \mathbf{t}\omega$,
(3) $\mathbf{t}(\star\omega) = \star\mathbf{n}(\omega)$,
(4) $\mathbf{n}(\star\omega) = \star\mathbf{t}(\omega)$,

(5) $d(\mathbf{t}\omega) = \mathbf{t}(d\omega)$,
(6) $\delta(\mathbf{n}\omega) = \mathbf{n}(\delta\omega)$.

Proof. (1) and (2) immediately follow from Definition 3.1. For the other assertions, observe that there are two ways of decomposing $\star\omega$:

$$\star(\omega) = \star(\mathbf{t}\omega) + \star(\mathbf{n}\omega), \qquad (\star\omega) = \mathbf{t}(\star\omega) + \mathbf{n}(\star\omega).$$

Hence, one has

$$\star(\mathbf{t}\omega) + \star(\mathbf{n}\omega) = \mathbf{t}(\star\omega) + \mathbf{n}(\star\omega),$$

moving terms,

$$\star(\mathbf{t}\omega) - \mathbf{n}(\star\omega) = \mathbf{t}(\star\omega) - \star(\mathbf{n}\omega).$$

Note that the next two conditions hold on both sides of the equation:

$$\begin{cases} \star(\mathbf{t}\omega), \mathbf{n}(\star\omega) & \text{involve} & dx_n \\ \mathbf{t}(\star\omega), \star(\mathbf{n}\omega) & \text{do not involve} & dx_n \end{cases}$$

Therefore,

$$\star(\mathbf{t}\omega) = \mathbf{n}(\star\omega), \qquad \mathbf{t}(\star\omega) = \star(\mathbf{n}\omega).$$

This disposes of (3) and (4).

It remains to proof (5) and (6), that is, we need to show that the exterior derivative commutes with $\mathbf{t}$, whereas the codifferential commutes with $\mathbf{n}$. The proof in the first case is straightforward. Recall that

$$\mathbf{t}\omega = i^*(\omega).$$

Applying the exterior derivative to both side, one has

$$d(\mathbf{t}\omega) = d(i^*(\omega)).$$

We now the exterior derivative commutes with the pullback, hence

$$d(i^*(\omega)) = i^*(d\omega) = \mathbf{t}(d\omega).$$

Therefore,

$$d(\mathbf{t}\omega) = \mathbf{t}(d\omega).$$

Lastly, using previous relations, we start by stating $d\mathbf{t}(\star\omega) = \mathbf{t}d(\star\omega)$. Applying the Hodge star on both sides:

$$\star d\mathbf{t}(\star\omega) = \star\mathbf{t}d(\star\omega).$$

Using identities previously demonstrated one gets:

$$\star d \star (\mathbf{n}\omega) = \mathbf{n} \star d(\star\omega).$$

Therefore, multiplying both sides of the equation above by $(-1)^{n(k+1)+1}$ and using the definition of the codifferential, yields

$$\delta(\mathbf{n}\omega) = \mathbf{n}(\delta\omega).$$

This concludes the proof. □

The differential and codifferential are related by the following formula. This formula is a consequence of Stokes' theorem and it is, in some sense, an integration by parts formula. Notably, it is a crucial identity for showing the orthogonality of the spaces involved in the Hodge decomposition theorem.

Proposition 3.3 (Green's formula)**.** *Let* $\alpha \in \Omega^k(M)$ *and* $\beta \in \Omega^{k+1}(M)$ *be differential forms on a manifold* M*. Then*

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_{\partial M} i^*(\alpha) \wedge i^*(\star\beta),$$

where i^* *denotes the pullback along the inclusion* $i : \partial M \to M$*.*

Proof. Consider the differential form $\alpha \wedge \star\beta$. Taking exterior derivative yields

$$d(\alpha \wedge \star\beta) = d\alpha \wedge \star\beta + (-1)^k \alpha \wedge d(\star\beta).$$

On the other hand, by Stokes's theorem, we have

$$\int_M d(\alpha \wedge \star\beta) = \int_{\partial M} i^*(\alpha \wedge \star\beta)$$

Putting these together, we get

$$\int_M d\alpha \wedge \star\beta + \int_M (-1)^k \alpha \wedge d(\star\beta) = \int_{\partial M} i^*(\alpha) \wedge i^*(\star\beta).$$

By the definition of the inner product, we thus have

$$\langle d\alpha, \beta\rangle + \int_M (-1)^k \alpha \wedge d(\star\beta) = \int_{\partial M} i^*(\alpha) \wedge i^*(\star\beta)$$

In other words, we get

$$\langle d\alpha, \beta\rangle = (-1)^{k+1} \int_M \alpha \wedge d(\star\beta) + \int_{\partial M} i^*(\alpha) \wedge i^*(\star\beta)$$

Now recall (Lemma 2.17) that

$$\star\star\,\eta = (-1)^{k(n-k)}\eta,$$

for any form $\eta \in \Omega^k(M)$, and

$$\delta\omega = (-1)^{n(k+1)+1} \star d \star \omega,$$

for any form $\omega \in \Omega^k(M)$. We shall use these identities to rewrite $(-1)^{k+1} \int_M \alpha \wedge d(\star\beta)$. Indeed,

$$(-1)^{k+1} \int_M \alpha \wedge d(\star\beta) = (-1)^{k+1} \int_M \alpha \wedge \left[(-1)^{k(n-k)} \star\star\, d(\star\beta)\right]$$

Regrouping and moving signs to the front we get

$$(-1)^{k+1} \int_M \alpha \wedge d(\star\beta) = (-1)^{k+1}(-1)^{k(n-k)} \int_M \alpha \wedge \star(\star d \star \beta)$$

Moreover, note that β is a $(k+1)$-form, so $\delta\beta = (-1)^{n(k+2)+1} \star d \star \beta = (-1)^{nk+1} \star d \star \beta$. Using this in the last equation above yields

$$(-1)^{k+1} \int_M \alpha \wedge d(\star\beta) = (-1)^{k+1}(-1)^{k(n-k)}(-1)^{nk+1} \int_M \alpha \wedge \star((-1)^{kn+1} \star d \star \beta)$$

More succintly,

$$(-1)^{k+1} \int_M \alpha \wedge d(\star\beta) = (-1)^{k+1}(-1)^{k(n-k)}(-1)^{nk+1} \int_M \alpha \wedge \star(\delta\beta)$$

Cleaning up signs we get

$$(-1)^{k+1} \int_M \alpha \wedge d(\star\beta) = (-1)^{k(1-k)} \int_M \alpha \wedge \star(\delta\beta) = \int_M \alpha \wedge \star(\delta\beta) = \langle \alpha, \delta\beta\rangle.$$

Finally, placing this information into the initial equality yields

$$\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle + \int_{\partial M} i^*(\alpha) \wedge i^*(\star\beta).$$

□

Remark 3.4. Recall that pullback commutes with wedge product and the exterior derivative, so one has $i^*(\omega \wedge \nu) = i^*(\omega) \wedge i^*(\nu)$, and $i^*(d\omega) = d(i^*\omega)$. Moreover, since $i^*(\omega) = \mathbf{t}(\omega)$, and the Hodge star $\star$ satisfies (cf. Proposition 3.2):

$$\mathbf{t}(\star\omega) = \star(\mathbf{n}\omega),$$

Green's formula can be rewritten as:

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle + \int_{\partial M} \mathbf{t}(\alpha) \wedge \star\mathbf{n}\beta.$$

If $\partial M = \emptyset$, then Green's formula illustrates the fact that d and δ are adjoint operators. This is no longer true when $\partial M \neq \emptyset$.

3.2. **Hodge decomposition for manifolds with boundary.** Let M be a smooth, compact, oriented n-dimensional manifold with boundary ∂M. Recall that $\Omega^k(M)$ denotes the space of smooth k-forms in M.

Definition 3.5. We denote by $\Omega^k_D(M)$ the space of all k-forms that satisfy the **Dirichlet boundary condition**, that is

$$\Omega^k_D(M) = \{\omega \in \Omega^k(M) \mid \mathbf{t}(\omega) = 0\}.$$

Similarly, we denote by $\Omega^k_N(M)$ the space of all k-forms that satisfy the **Neumann boundary condition**, that is

$$\Omega^k_N(M) = \{\omega \in \Omega^k(M) \mid \mathbf{n}(\omega) = 0\}.$$

In virtue of Proposition 3.2, we have that if $\omega \in \Omega_D(M)$, then $\star\omega \in \Omega_N(M)$, and vice versa. That is, the Hodge star $\star$ takes Dirichlet forms into Neumann forms and vice versa.

By Proposition 3.2, the space of differential forms with Neumann boundary condition can be also characterized as follows.

Lemma 3.6. $\Omega^k_N(M) = \{\omega \in \Omega^k(M) \mid i^*(\star\omega) = 0\}$. □

Remark 3.7. Let ω be a k-form that satisfies the Dirichlet boundary condition. Then $\mathbf{t}\omega$, the tangent component of ω, is zero. For this reason, a form satisfying the Dirichlet boundary condition is often referred to as **perpendicular or normal to the boundary** ∂M.

Let ω be a k-form that satisfies the Neumann boundary condition. Then $\mathbf{n}\omega$, the normal component of ω, is zero. Hence, a form satisfying the Neumann boundary condition is usually called **tangent to the boundary** ∂M.

If $\partial M = \emptyset$, then we set $\Omega^k_D(M) = \Omega^k_N(D) = \Omega^k(M)$. Using this convention, all the arguments given below apply to the case of manifolds without boundary as well.

Definition 3.8. For each k, we define the following subspaces of $\Omega^k(M)$.

- The space of **exact forms with Dirichlet boundary condition**:

$$d\Omega^{k-1}_D(M) = \mathcal{E}^k_D(M) = \{\xi \in \Omega^k(M) \mid \xi = d\alpha,\ \alpha \in \Omega^{k-1}_D(M)\}$$

- The space of **co-exact forms with Neumann boundary condition**:

$$\delta\Omega^{k+1}_N(M) = c\mathcal{E}^k_N(M) = \{\nu \in \Omega^k(M) \mid \nu = \delta\beta,\ \beta \in \Omega^{k+1}_N(M)\}$$

- The space of **harmonic k-fields** (i.e. the closed and co-closed k-forms):

$$\mathcal{H}^k(M) = CcC^k(M) = \{\gamma \in \Omega^k(M) \mid d\gamma = 0,\ \delta\gamma = 0\}.$$

Note that the definitions above are required to account for the presence of the boundary.

Throughout the rest of this section, the manifold M shall be considered fixed, so to simplify notation we shall, when needed, omit the reference to M in the notation and usage of the subspaces of $\Omega^k(M)$, and simply write $d\Omega^{k-1}$, $\delta\Omega^{k+1}$, $\mathcal{H}^k$ and so on.

Remark 3.9. When $\partial M \neq \emptyset$, the nomenclature harmonic field for the k-forms in $\mathcal{H}^k$ is due to Kodaira. This is to distinguish them from the harmonic forms, i.e. those forms ω that satisfy $\Delta\omega = 0$, where $\Delta = d\delta + \delta d$ is the Hodge-Laplacian. On a manifold with boundary, a harmonic field is certainly a harmonic form, but the converse is not true in general. See [CDTGM, Example p. 2]. In other words, the presence of the boundary plays an important role in this setting. When the boundary ∂M is empty, the concepts of harmonic field and harmonic form are equivalent.

The first main result in this setting is the following.

Theorem 3.10 (Hodge-Morrey Decomposition)**.** *Let M be a smooth, compact, oriented n-dimensional manifold with boundary. Let $\omega \in \Omega^k(M)$. Then there are differential forms $\alpha \in \Omega_D^{k-1}(M)$, $\beta \in \Omega_N^{k+1}(M)$, and $\gamma \in \mathcal{H}^k(M)$ such that*

$$\omega = d\alpha + \delta\beta + \gamma.$$

Furthermore, the differential forms $d\alpha$, $\delta\beta$, and γ are mutually L^2-orthogonal, and so are uniquely determined. That is, there is an L^2-orthogonal decomposition

$$\Omega^k = d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathcal{H}^k.$$

Proof. We shall only prove the orthogonality statement. The proof that these three subspaces indeed span $\Omega^k(M)$ relies on methods from analysis on Hilbert spaces, and go beyond the scope of this paper, so we do not include it here. The interested reader is invited to consult [S, Lemma 2.4.3 (a)] for a proof of the span statement. For compact domains in $\mathbb{R}^3$, the span statement can be proved in a more elementary way, using solutions of the Laplace equation with Dirichlet and Neumann boundary conditions, see [CDTG] for details.

First, we prove that the space $\mathcal{E}_D^k$ is orthogonal to cC^k (space of co-closed k-forms). Indeed, let $\eta \in \mathcal{E}_D^k$. Then there is $\xi \in \Omega^{k-1}(M)$ such that $\eta = d\xi$ and $\mathbf{t}(\xi) = 0$. Now let $\theta \in cC^k$. By Green's formula (Proposition 3.3 and Remark 3.4) we get

$$\langle \eta, \theta \rangle = \langle d\xi, \theta \rangle = \langle \xi, \delta\theta \rangle + \int_{\partial M} \mathbf{t}(\xi) \wedge \star\mathbf{n}(\theta).$$

Since $\mathbf{t}(\xi) = 0$ and $\delta\theta = 0$, we thus conclude

$$\langle \eta, \theta \rangle = 0.$$

Second, let us show that the space $c\mathcal{E}_N^k$ is orthogonal to C^k (space of closed k-forms). Indeed, let $\eta \in c\mathcal{E}^k$. Then there is $\omega \in \Omega^{k+1}(M)$ such that $\eta = \delta\omega$ and $\mathbf{n}(\omega) = 0$. The latter condition yields, in particular, $\star\mathbf{n}(\omega) = 0$.

So if we take $\theta \in C^k$, then Green's formula (Proposition 3.3 and Remark 3.4) implies

$$\langle d\theta, \omega \rangle = \langle \theta, \delta\omega \rangle + \int_{\partial M} \mathbf{t}(\theta) \wedge \star\mathbf{n}(\omega).$$

Since $d\theta = 0$ and $\star\mathbf{n}(\omega) = 0$, we conclude

$$\langle \theta, \eta \rangle = \langle \theta, \delta\omega \rangle = 0.$$

Third, note that $\mathcal{E}_D^k$ is a subset of C^k. Since, by the second step of the proof, we already know that $c\mathcal{E}_N^k$ is orthogonal to C^k, then, in particular, $c\mathcal{E}_N^k$ is orthogonal to $\mathcal{E}_D^k$.

Finally, the harmonic fields $\mathcal{H}^k$ are both closed and co-closed, hence from the previous steps it is clear that this subspace is orthogonal to both $\mathcal{E}_D^k$ and $c\mathcal{E}_N^k$.

In summary, the three subspaces $\mathcal{E}_D^k$, $c\mathcal{E}_N$, and $\mathcal{H}^k$ are mutually orthogonal in Ω^k. This concludes the proof. □

Remark 3.11. Theorem 3.10 reduces to the classical Hodge decomposition theorem. In particular, if M is compact and boundaryless, then $\mathcal{H}^k(M) \simeq H^k_{dR}(M)$. See Theorem 3.18. In particular, in case M has empty boundary, then $H^k_{dR}(M)$ has a unique harmonic representative. Furthermore, this harmonic representative minimizes the norm in a given cohomology class. See [S] for further details.

The spaces of harmonic fields $\mathcal{H}^k$ are in general infinite dimensional (unless M has empty boundary, by the previous remark). As such, these spaces are too big to directly encode information coming from the geometry of M (i.e. the de Rham cohomology of M). Nevertheless, the spaces of harmonic fields come equipped with some finite dimensional subspaces that do encode geometric information of M. To be more precise, we need the following.

Definition 3.12. Let $\mathcal{H}_D^k$ denote the space of harmonic fields with Dirichlet boundary condition. Namely,

$$\mathcal{H}_D^k = \{\omega \in \mathcal{H}^k(M) \,|\, \mathbf{t}(\omega) = 0\}.$$

Similarly, let $\mathcal{H}_N^k$ denote the space of harmonic fields with Neumann boundary condition. That is,

$$\mathcal{H}_N^k = \{\omega \in \mathcal{H}^k(M) \,|\, \mathbf{n}(\omega) = 0\}.$$

It is worth noting that the boundary conditions on a harmonic field refer to conditions on the field itself, whereas the boundary conditions on a exact or co-exact form in Theorem 3.8 apply to its primitive or "potential" form.

By [S, Theorem 2.2.2] the spaces in Definition 3.12 are finite dimensional.

Remark 3.13. The Hodge star defines an isomorphism (sometimes called Hodge duality)

$$\star : \Omega^k(M) \to \Omega^{n-k}(M)$$

which in turn induces an isomorphism between $\mathcal{H}_D^k$, the space of Dirichlet fields, and $\mathcal{H}_N^{n-k}$, the space of Neuman fields. These finite dimensional subspaces of $\Omega^k(M)$ and are concrete realizations of the de Rham cohomology groups of M inside $\Omega^k(M)$, and as such they reflect many of the topological properties of M. See Theorem 3.18.

Using the spaces in Definition 3.12, it is possible to further decompose the space of harmonic fields. Indeed, as they are finite dimensional subspaces of $\mathcal{H}^k$ (and hence they are closed), a straightforward way to do so is by taking orthogonal complements:

$$\mathcal{H}^k = \mathcal{H}_N^k \oplus (\mathcal{H}_N^k)^\perp$$
$$\mathcal{H}^k = \mathcal{H}_D^k \oplus (\mathcal{H}_D^k)^\perp.$$

Nevertheless, an immediate related question is: can we recognize explicitly the orthogonal complements of $\mathcal{H}_N^k$ and $\mathcal{H}_D^k$ inside $\mathcal{H}^k$? The answer is given by the Friedrichs decomposition (see Theorem 3.15 below). To state it, we need some additional terminology.

Definition 3.14. Notation as above, we define

$$\mathcal{H}_{co}^k = \{\omega \in \mathcal{H}^k(M) \mid \omega = \delta\gamma, \text{ for some } \gamma \in \Omega^{k+1}(M)\},$$
$$\mathcal{H}_{ex}^k = \{\omega \in \mathcal{H}^k(M) \mid \omega = d\gamma, \text{ for some } \gamma \in \Omega^{k+1}(M)\}$$

the spaces of co-exact and exact harmonic fields, respectively. See [S, Definition 2.2.1].

Theorem 3.15 (Friedrichs decomposition). *On a compact manifold M with boundary δM, the spaces of harmonic fields $\mathcal{H}^k$ can respectively be decomposed into*

$$\mathcal{H}^k = \mathcal{H}^k_D \oplus \mathcal{H}^k_{co}$$

and

$$\mathcal{H}^k = \mathcal{H}^k_N \oplus \mathcal{H}^k_{ex}$$

For a proof see [S, Theorem 2.4.8].

The previous theorem identifies precisely the orthogonal complements of the finite dimensional subspaces $\mathcal{H}^k_D$ and $\mathcal{H}^k_N$ inside $\mathcal{H}^k$.

In summary, combining Theorems 3.10 and 3.15 yields the following.

Theorem 3.16 (Friedrichs-Hodge-Morrey decomposition). *On a manifold with boundary the space of k-forms admits the following L^2-orthogonal decompositions:*

$$\Omega^k = d\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N \oplus \mathcal{H}^k_{ex} \oplus \mathcal{H}^k_N.$$

Similarly,

$$\Omega^k = d\Omega^{k-1}_D \oplus \delta\Omega^{k+1}_N \oplus \mathcal{H}^k_{co} \oplus \mathcal{H}^k_D.$$

□

Remark 3.17. This result implies the **Helmholtz decomposition** on manifolds with boundary. Recall that this states: each differential form ω can be uniquely split into a co-exact part $\omega_{co} \in \delta\Omega^{k+1}_N(M) \oplus \mathcal{H}_{co}(M)$ and a closed part $\omega_{cl} \in d\Omega^{k-1}_D(M) \oplus \mathcal{H}^k_D$, the latter with vanishing tangential component [S]. Likewise, it can also be split into an exact part and a co-closed part with vanishing normal component. In a more classical language, this decomposition states that any vector field F on a compact domain $M \subset \mathbb{R}^3$ can be written as $G + K$ where $\mathbf{rot}(G) = \vec{0}$, $\mathbf{div}(K) = 0$ and K is tangent to the boundary of M. If we make the identification of vector fields with one forms in $\mathbb{R}^3$, this decomposition is analogous in particular to the second interpretation of Friedrichs-Hodge-Morrey decomposition that split each differential form into a exact part and a co-closed part with vanishing normal component. Moreover, as we shall see in the next section, the summand G is a gradient vector field and, in the nomenclature of [CDTG], the summand K is called a *knot* on M.

3.3. **Geometric side of the Hodge decomposition.** Some of the summands in the Hodge decomposition of a k-form reflect particular geometric properties of the domain where the form is defined. In fact, we shall see below that the harmonic fields with Dirichlet and Neumann boundary conditions correspond to the relative and global de-Rham cohomology groups of M. Recall the notation from Subsection 2.7.

Theorem 3.18. *Let M be a smooth, compact, oriented, n-dimensional manifold with boundary ∂M. The following hold.*

1. *The k-cohomology group of the de Rham complex $(\Omega^*(M), d)$ is isomorphic to $\mathcal{H}^k_N(M)$. In other words, $\mathcal{H}^k_N(M) \simeq H^k_{dR}(M)$.*
2. *The k-cohomology group of the complex $(\Omega^*(M), \delta)$ is isomorphic to $\mathcal{H}^k_D(M)$.*

Proof. Simply use the Hodge-Morrey-Friedrichs decomposition (Theorem 3.16) to show that the orthogonal complement to the exact forms inside the closed forms is isomorphic to $\mathcal{H}^k_N(M)$. Thus $H^k_{dR}(M) \simeq \mathcal{H}^k_N(M)$.

Likewise, using the orthogonality of the Hodge-Morrey-Friedrichs decomposition (Theorem 3.16) it is clear that the orthogonal complement to the co-exact forms within the co-closed forms is isomorphic to $\mathcal{H}^k_D(M)$. Hence, $\mathbf{H}^k(M, \delta) \simeq \mathcal{H}^k_D(M)$. □

Remark 3.19. Note that δ respects the Neumann condition. Indeed, if $\omega \in \Omega^k_N(M)$, then $\delta\omega \in \Omega^{k-1}_N(M)$, because $\mathbf{n}(\delta\omega) = \delta(\mathbf{n}\omega) = 0$ (Proposition 3.2). That is, we get a complex $(\Omega^k_N(M), \delta)$. This complex is sometimes called the absolute chain complex of M. The cohomology of this complex is denoted $\mathbf{H}^k_a(M)$. Using Theorem 3.16 and arguing as in the proof of the previous result, one obtains that $\mathcal{H}^k_N(M) \simeq \mathbf{H}^k_a(M)$. Thus $H^k_{dR}(M) \simeq \mathcal{H}^k_N(M) \simeq \mathbf{H}^k_a(M)$. See [S, Corollary 2.6.2]

It is worth noting that d respects the Dirichlet condition. Indeed, if $\omega \in \Omega^k_D(M)$, then $d\omega \in \Omega^{k+1}_D(M)$, for $\mathbf{t}(d\omega) = d(\mathbf{t}\omega) = 0$, by Proposition 3.2. Thus, we get a complex $(\Omega^k_D(M), d)$, the relative chain complex of M. The cohomology of this complex corresponds to $\mathbf{H}^k_r(M)$, the relative de Rham cohomology of M, denoted $H^k_{dR}(M, \partial M)$.

Lemma 3.20. *Notation being as above, $\mathbf{H}^k_r(M)$ and $\mathcal{H}^k_D(M)$ are isomorphic.*

Proof. Use Theorem 3.16 to show that $\mathcal{H}^k_D(M)$ also computes relative cohomology. See [S, Corollary 2.6.2] for details. □

Consequently, we have

$$\mathbf{H}^k(M, \delta) \simeq H^k_{dR}(M, \partial M) \simeq \mathcal{H}^k_D(M),$$
$$\mathbf{H}^k(M, d) \simeq H^k_{dR}(M) \simeq \mathcal{H}^k_N(M).$$

In view of the de Rham isomorphism (Theorem 2.21), the previous two isomorphisms imply that $\mathcal{H}^k_N(M)$ and $\mathcal{H}^k_D(M)$ are realizations of the absolute cohomology $H^k(M, \mathbb{R})$ and the relative cohomology $H^k(M, \partial M, \mathbb{R})$, respectively.

Moreover, the Hodge duality (Remark 3.13) yields the following isomorphism.

Corollary 3.21. *The Hodge star induces the following isomorphism*

$$\star : H^k_{dR}(M) \simeq H^{n-k}_{dR}(M, \partial M).$$

□

This latter isomorphism can be thought of as a differential form version of the Poincaré duality isomorphism for manifolds wih boundary.

It is worth noting that $\mathcal{H}^k_N(M)$ and $\mathcal{H}^k_D(M)$ only intersect at the origin. Intuitively, this says that a harmonic field that vanishes on the boundary must vanish everywhere (certainly, this holds for harmonic 0-fields, by the maximum principle).

Theorem 3.22. [S, Theorem 3.4.4] *Let M be a manifold with non-empty boundary ∂M. If a harmonic field $\gamma \in \mathcal{H}^k(M)$ vanishes on the boundary, then $\gamma = 0$ on M. In other words,*

$$\mathcal{H}^k_D(M) \cap \mathcal{H}^k_N(M) = \{0\}.$$

□

In view of the previous result, one may wonder if the orthogonal decomposition of Theorem 3.16 could be further split into a five summand decomposition, so as to have both $\mathcal{H}^k_D(M)$ and $\mathcal{H}^k_N(M)$ appearing as direct summands. However, this is, in general, not possible, because for general manifolds the spaces $\mathcal{H}^k_N(M)$ and $\mathcal{H}^k_D(M)$ are not orthogonal. See [Sh]. Remarkably, for compact domains in $\mathbb{R}^3$, we shall see that these two spaces are indeed orthogonal. See Lemma 3.24.

In the general case, the Hodge-Morrey and Friedrichs decompositions can be merged into a single decomposition

Theorem 3.23. [P, Identity 2.16] *Notation as above, we have*

$$\Omega^k = d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus d\Omega^{k-1} \cap \delta\Omega^{k+1} \oplus (\mathcal{H}_D^k + \mathcal{H}_N^k),$$

where the sum $\mathcal{H}_N^k + \mathcal{H}_D^k$ is a direct, but in general not an L^2-orthogonal sum. □

We shall see that for compact domains in $\mathbb{R}^3$ the latter decomposition yields a five-summand decomposition of vector fields defined on such domains. For this, an additional result is needed.

Lemma 3.24. [P, Lemma 2.4.6] *Let M be a bounded domain in $\mathbb{R}^3$. Then $\mathcal{H}_D^k$ and $\mathcal{H}_N^k$ are L^2-orthogonal.* □

Summarizing we obtain the Hodge decomposition theorem for compact domains in $\mathbb{R}^3$.

Theorem 3.25. *Let M be a bounded domain in $\mathbb{R}^3$. Then for $k = 1, 2$ there is an L^2-orthogonal decomposition*

$$\Omega^k = d\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus d\Omega^{k-1} \cap \delta\Omega^{k+1} \oplus \mathcal{H}_D^k \oplus \mathcal{H}_N^k.$$

□

In the next section we shall identify the summands appearing in Hodge decomposition (Theorem 3.25) with those appearing in the vector-field version of this decomposition, due to Cantarella, DeTurck, and Gluck [CDTG].

4. Hodge decomposition for compact domains in $\mathbb{R}^3$

Let M be a compact regular domain in 3-space with smooth boundary ∂M. Let $\mathrm{VF}(M)$ be the set of all smooth vector fields defined on M. The space $\mathrm{VF}(M)$ is an infinite-dimensional (real) vector space. On the space $\mathrm{VF}(M)$, we consider the L^2-inner product given by

$$\langle V, W\rangle = \int_M V \cdot W \, d(vol)$$

Since M has dimension three, we can identify $\mathrm{VF}(M)$ with both $\Omega^1(M)$ and $\Omega^2(M)$. The first identification is quite natural, since $\mathrm{VF}(M)$ and $\Omega^1(M)$ are dual vector spaces. The second identification follows from the Hodge-star isomorphism $\star : \Omega^1(M) \simeq \Omega^2(M)$. Thus, in the particular case of solids or compact domains in $\mathbb{R}^3$, bear in mind that vector fields can be considered as either 1-forms or 2-forms (in the language of [P], these are the vector proxies of a vector field). Certainly, this is the point of view adopted in our first encounter with vector calculus, line integrals and surface integrals. As seen in the previous section, both $\Omega^1(M)$ and $\Omega^2(M)$ come equipped with a natural L^2-inner product:

$$\langle \alpha, \beta\rangle = \int_M \alpha \wedge \star\beta,$$

where $\alpha, \beta \in \Omega^k(M)$, with $k = 1, 2$. Under the corresponding identification, the products displayed above are equal. See e.g. [BM] or [MRA].

In what follows we shall, unless otherwise stated, identify $\Omega^1(M)$ with $\mathrm{VF}(M)$.

In the context of vector calculus, [CDTG] obtained a Hodge decomposition of the space $\mathrm{VF}(M)$ as a direct sum of five mutually orthogonal subspaces:

$$\mathrm{VF}(M) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

where

$\mathrm{FK} = \{V \in \mathrm{VF}(M) \,|\, \nabla\cdot V = 0,\ V\cdot\vec{n} = 0,\ \text{all interior fluxes are } 0\}$

$\mathrm{HK} = \{V \in \mathrm{VF}(M) \,|\, \nabla\cdot V = 0,\ \nabla\times V = 0,\ V\cdot\vec{n} = 0\}$

$\mathrm{CG} = \{V \in \mathrm{VF}(M) \,|\, V = \nabla\varphi \text{ for some } \varphi \in C^\infty(M),\ \nabla\cdot V = 0,\ \text{all boundary fluxes are } 0\}$

$\mathrm{HG} = \{V \in \mathrm{VF}(M) \,|, V = \nabla\varphi,\ \text{for some } \varphi \in C^\infty(M),\ \nabla\cdot V = 0,\ \varphi \text{ is locally constant on } \partial\mathrm{M}\}$

$\mathrm{GG} = \{V \in \mathrm{VF}(M) \,|\, V = \nabla\varphi,\ \text{for some } \varphi \in C^\infty(M),\ \varphi|_{\partial M} = 0\}$

These spaces are called *fluxless knots*, *harmonic knots*, *curly gradients*, *harmonic gradients*, and *grounded gradients*, respectively [CDTG]. Here $\vec{n}$ is the unit outward normal vector defined on ∂M.

The purpose of present section is to show that this decomposition is a realization of Theorem 3.25, by identifying each subspace above with one of the subspaces appearing in Theorem 3.25. This provides a differential form version of the results appearing in [CDTG]. We claim that the identifications are as follows:

$\mathrm{FK} = \delta\Omega^2 = \{\delta\beta \,|\, \beta \in \Omega^2(M) \text{ and } \mathbf{n}(\beta) = 0\}$

$\mathrm{HK} = \mathcal{H}^1_N = \{\gamma \in \Omega^1 \,|\, d\gamma = 0,\ \delta\gamma = 0,\ \mathbf{n}(\gamma) = 0\}$

$\mathrm{CG} = d\Omega^0 \cap \delta\Omega^2 = \{\gamma \in \Omega^1(M) \,|\, \gamma = df = \delta\beta,\ \text{for some } f \in \Omega^0(M) \text{ and } \beta \in \Omega^2(M)\}$

$\mathrm{HG} = \mathcal{H}^1_D = \{\gamma \in \Omega^1(M) \mid d\gamma = 0,\ \delta\gamma = 0,\ \mathbf{t}(\gamma) = 0\}$

$\mathrm{GG} = d\Omega^0 = \{df \mid f \in \Omega^0 \text{ and } \mathbf{t}(f) = 0\}$

It is worth noting that the 2-form β in FK can be chosen so that $d\beta = 0$. This is obtained by using Theorem Hodge-Morrey and arguing as in the proof of Theorem 3.18. Observe also that $\delta\beta$ is unique (for a given $\omega \in \Omega^1$), even though β is not necessarily unique (because we can take $\beta + (co-exact\,form)$). Certainly the above identifications can be readily made by using the *musical isomorphisms* [L], [P]. Nevertheless, instead of introducing new terminology, we prefer a more self-contained and explicit approach.

Henceforth we recover the geometric properties of the subspaces appearing in Theorem 3.25. We shall see that even though the geometric properties seem hidden in the differential-form version of the Hodge decomposition, they are in fact concisely encoded in the language of exact and co-exact forms and the Dirichlet and Neumann boundary conditions.

4.1. **Identifying FK and $\delta\Omega^2$.** Recall that the subspace of fluxless knots FK, in vector notation, is defined by:

$\mathrm{FK} = \{V \in \mathrm{VF}(M) \,|\, \nabla\cdot V = 0,\ V\cdot\vec{n} = 0,\ \text{all interior fluxes are } 0\}$,

while in the language of differential forms, this information is encoded in:

$\mathrm{FK} = \delta\Omega^2 = \{\delta\beta \,|\, \beta \in \Omega^2(M) \text{ and } \mathbf{n}(\beta) = 0\}$,

so that $\alpha = \delta\beta$ is the differential form analogue of the fluxless knot V, with basis $(dx, dy.dz)$ (on $\mathbb{R}^3$). In fact, we say they are dual because the i-th coefficient of α corresponds to the i-th component of the vector field V. Namely,

$$\alpha = V_1 dx + V_2 dy + V_3 dz.$$

(1) Let's start by showing $\nabla\cdot V = 0$. Recall that the δ operator applied to a 1-form gives back, up to sign, the expression of *div*. Hence,

$$\delta\alpha = -\nabla\cdot V.$$

If we write α as $\delta\beta$, it becomes evident that $div(V) = 0$ since δ (the codifferential) is nilpotent of degree 2, just like the exterior derivative:

$$\delta\alpha = \delta\delta\beta = 0$$

$$\therefore \nabla \cdot V = 0.$$

(2) We will show now that $V \cdot \vec{n} = 0$, it is, V is tangent to the boundary. In the language of differential forms, it is expressed as $\mathbf{n}(\alpha) = 0$, or literally, the normal component on ∂M is zero. We know δ respects the Neumman boundary condition, it is:

$$\mathbf{n}\delta = \delta\mathbf{n}$$

In other words, δ commutes with $\mathbf{n}$. Take into account that $\mathbf{n}(\beta) = 0$ and $\alpha = \delta\beta$, hence

$$\mathbf{n}\alpha = \mathbf{n}\delta\beta = \delta\mathbf{n}\beta = 0$$

Therefore,

$$\mathbf{n}\alpha = 0$$

Such condition only holds on ∂M. We have shown α is tangent to the boundary of M.

(3) Finally, the condition that all interior fluxes are zero is not explicit in exterior notation, but is encoded on $\mathbf{n}\beta = 0$, as we will see. We have to show that the flux of α through any cross-sectional surface Σ on M is zero. We define the flux of $\alpha = \delta\beta$ through Σ as

$$\int_\Sigma \star\alpha$$

Working out the integral, we have

$$\int_\Sigma \star\alpha = \int_\Sigma \star\delta\beta = \int_\Sigma \star(\star d \star \beta) = \int_\Sigma d \star \beta$$

We have cleaned up the signs already in the calculations (actually, the sign is not relevant as we want the result to be zero). Having the integral of an exact forms suggests to apply the generalized Stokes' theorem, which we do next:

$$\int_\Sigma d \star \beta = \int_{\partial\Sigma} \star\beta$$

In order to work out the integral, given that $\partial\Sigma \in \partial M$, we use the pullback along the inclusion map i.

$$\int_{\partial\Sigma} \star\beta = \int_{\partial\Sigma} i^*(\star\beta).$$

In fact, we are using the generalized Stokes' Theorem, 2.19. By the definition of the operator $\mathbf{t}$ we have:

$$\mathbf{t}(\star\beta) = i^*(\star\beta)$$

Recall now that $\mathbf{t}(\star\beta) = \star(\mathbf{n}\beta)$. In short, we can write $\star(\mathbf{n}\beta)$ to replace $i^*(\star\beta)$

$$\int_{\partial\Sigma} i^*(\star\beta) = \int_{\partial\Sigma} \star(\mathbf{n}\beta)$$

Finally, we use $\mathbf{n}\beta = 0$

$$\int_{\partial\Sigma} \star(\mathbf{n}\beta) = 0$$

$$\therefore \int_\Sigma \star\alpha = 0$$

We have shown that the flux of α through any cross-sectional surface inside M is zero.

4.2. **Identifying HK and $\mathcal{H}^1_N$.** First, we will show the geometric properties of the harmonic knots are encoded in the exterior notation of the subspace.

We state the subspace of harmonic knots in vector calculus and differential forms, respectively, as:

$\mathrm{HK} = \{V \in \mathrm{VF}(M) \,|\, \nabla \cdot V = 0,\ \nabla \times V = 0,\ V \cdot \vec{n} = 0\}$

$\mathrm{HK} = \mathcal{H}^1_N = \left\{\gamma \in \Omega^1 \,|\, d\gamma = 0,\ \delta\gamma = 0,\ \mathbf{n}(\gamma) = 0\right\}$

We define γ to be the exterior form of the vector field V, so that:

$$\gamma = V_1 dx_1 + V_2 dx_2 + V_3 dx_3,$$

for $V = (V_1, V_2, V_3)$ to be a harmonic knot.

(1) Since the codifferential applied to γ gives back $-div(V)$, we write, using $\delta\gamma = 0$:

$$\delta\gamma = -\nabla \cdot V = 0,$$

$$\therefore \nabla \cdot V = 0.$$

(2) We now show that $rot(V) = 0$ by using $d\gamma = 0$. As γ is a 1-form, the exterior derivative gives back a 2-form whose three coefficients are the components of the rotational of V. Therefore, $d\gamma = 0$ implies $rot(V) = \nabla \times V = 0$.

(3) Lastly, the condition $V \cdot \vec{n} = 0$, which holds only on ∂M, means V is tangent to the boundary. The Neumman boundary condition says precisely the same by imposing $\mathbf{n}(\gamma) = 0$, so if the normal component is zero, then the field can be only tangent to the boundary.

We will now show the abundance of the subspace of harmonic knots. One can think that harmonic fields can only be zero because of the many conditions they have, this is why we make sure they exist by demonstrating their abundance in the next steps. Furthermore, we claim they encode geometric information of the topology of the domain of definition.

Lemma 4.1. *Let M be a compact domain in $\mathbb{R}^3$. Then $H^1(M) \simeq H^2(M, \partial M) \simeq \mathbb{R}^g$, where g is the genus of ∂M.*

Recall that $g = \sum_{i \in I} g_i$, where I indexes the connected components of ∂M and g_i denotes the genus of the i-th connected component of ∂M. For a proof see [CDTG, Lemma 2] or [E, Proposition 4.12].

Given that $\mathcal{H}^1_N(M)$ is isomorphic to $H^1_{dR}(M)$ (Theorem 3.18), the de Rham isomorphism (Theorem 2.21) yields

Lemma 4.2. [CDTG, Lemma 2]

$$\mathcal{H}^1_N(M) \simeq \mathbb{R}^g.$$

In other words, the harmonic knots are abundant in M. Indeed, $\mathrm{HK} \simeq H^1_N(M)$ and so, it is a finite-dimensional subspace of $\mathrm{VF}(M)$ that consists of vector fields that are divergence-free, curl-free and tangent to the boundary. Moreover, HK reflects the topology of M: the number of generators of HK depends on the genus of ∂M, and therefore, on the dimension of $H_1(M)$ (the vector space of equivalence classes of oriented loops, where two loops are declared equivalent if their difference bounds a surface in M).

Example 4.3. For the case of the solid torus, HK is one dimensional, whereas for the solid sphere, HK is zero.

In this context, how do we generate explicitly harmonic knots from the knowledge of the genus g? The method is developed in the proof of [CDTG, Lemma 2] (cf. [E, Lemma 7.1]). There the authors make extensive use of the Biot-Savart formula for vector fields and the homology and cohomology on three-space. We shall illustrate their method in the case when $g = 1$, i.e. when M is a solid torus. The reader could consult [CDTG] for more details.

Let M be a solid torus of revolution, C be the closed curve that goes all the way around the torus, and C' a circle that passes through the hole of M. Now run a current I through C'. This generates a magnetic field defined on $\mathbb{R}^3$ except on C' itself. Let us denote by B the restriction of this magnetic field to M. Such magnetic field has the following properties:

$$\nabla \cdot B = 0$$

$$\nabla \times B = 0$$

$$\int_C B \cdot ds = I$$

Next, we modify B slightly, so as to make it tangent to the boundary of M. We do so by extracting from B an appropriate gradient vector field. This gradient vector field should not alter the already good properties of B, i.e. we need $\nabla\varphi$ such that $B - \nabla\varphi$ is a divergence-free vector field. Thus, we are looking for a smooth function φ such that $\nabla \cdot \nabla\varphi = \Delta\varphi = 0$. That is, φ should be a harmonic function. Nevertheless, we need to annihilate the normal component of B. Thus, φ should, if any, be chosen so that $(B - \nabla\varphi) \cdot \vec{n} = 0$. That is, we need a smooth function φ on M that satisfies the following conditions:

$$\Delta\varphi = 0$$

$$\frac{\partial\varphi}{\partial\mathbf{n}} = \nabla \cdot \vec{n} = B \cdot \vec{n}.$$

This is a Neumann boundary problem. Since B is divergence-free and defined everywhere on M, we have $\int_{\partial M} B \cdot \vec{n} d(area) = 0$. Thus a solution φ to the Neumann problem given above exists.

With such a solution at our disposal, now consider $V = B - \nabla\varphi$. We claim that $V \in HK$. Indeed, since

$$\nabla \cdot V = 0,\ \nabla \times V = 0,\ V \cdot n = 0,\ \int_C V \cdot ds = I$$

In particular, V is nonzero. In [CDTG], this technique is extended to more general domains. The conclusion is that the number of generators of the space HK is equal to the dimension of the absolute homology vector space $H_1(M)$. Furthermore,

$$\mathrm{HK} \cong H_1(M) \cong H_2(M, \partial M) \cong \mathbb{R}^g,$$

where g is the genus of ∂M.

4.3. **Identifying CG and $d\Omega^0 \cap \delta\Omega^2$.** We will show the equivalence of the proposed definition of the subspace of curly gradient using differential forms and the definition in the context of vector calculus. In the context of vector calculus:

CG $= \{V = \nabla\varphi,\ \nabla \cdot V = 0,$ all boundary fluxes are $0\}$

And in the language of differential forms, according to the decomposition proposed in the theorem 2.15, the curly gradients could also be defined as:

$$\mathrm{CG} = d\Omega^0(M) \cap \delta\Omega^2(M) = \big\{\gamma \in \Omega^1(M) \,|\, \gamma = df = \delta\beta,\ \text{for some } f \in \Omega^0(M) \text{ and } \beta \in \Omega^2(M)\big\}$$

(1) Since we are making the identification of vector fields with one forms and γ is exact:

$$V = \nabla\varphi \to \gamma = df$$

(2) Since $\gamma = \delta\beta$ and δ is nilpotent of degree 2:

$$div(V) \to \delta\gamma = \delta\delta\beta = 0$$

(3) We can notice that the geometric condition in the vector calculus definition, all boundary fluxes are zero, does not appear in the differential forms context. However, as we shall see, this condition is already implied by $d\Omega^0 \cap \delta\Omega^2$, particularly $\delta\Omega^2$.

Let ϕ_{ij} be the flux on ∂M_{ij},where ∂M_{ij} refers to the j-th connected component of the boundary of the i-th connected component of M. Then:

$$\phi_{ij} = \int_{\partial M_{ij}} i^* \star \gamma = \int_{\partial M_{ij}} i^* \star\star\, d \star \beta = \int_{\partial M_{ij}} (-1)\, d\, i^* \star \beta$$

and by the generalized Stokes' theorem,

$$\int_{\partial M_{ij}} (-1)\, d\, i^* \star \beta = \int_{\partial\partial M_{ij}} (-1)\, i^* \star \beta = 0.$$

which is zero because ∂M_{ij} has no boundary.

4.4. **Identifying HG and $\mathcal{H}^1_D$.** First, we will show the equivalence of the proposed definition. In the context of vector calculus, the subspace of harmonic gradients, HG, is defined as:

HG $= \{V \in \mathrm{VF}(M)\,|V = \nabla\varphi,$ for some $\varphi \in C^\infty(M),\ \nabla \cdot V = 0,\ \varphi$ is locally constant on ∂M$\}$

And in the language of differential forms, according to the decomposition proposed in the theorem 2.15, the harmonic gradients could also be defined as:

HG $= \mathcal{H}^1_D(M) = \left\{\gamma \in \Omega^1(M) \mid d\gamma = 0,\ \delta\gamma = 0,\ \mathbf{t}(\gamma) = 0\right\}$

(1) Upon initial inspection, the condition $\gamma = df$, which is the equivalent of $V = \nabla\varphi$, is not stated in the proposed definition. However, the conditions are enough to prove this property. As seen in Section 2, if we show that the integral over any closed path on M for a differential form γ is zero, we can assert that γ is exact. To do so, we need to rely on the homology of the space. According to [CDTG, p.424], it is possible to find a closed path C' on ∂M so:

$$\int_C i^*\gamma = \int_{C'} i^*\gamma,$$

where C is any closed path on M. Therefore, for any harmonic gradient γ, it is true that:

$$\int_C i^*\gamma = \int_{C'} i^*\gamma = 0$$

because $\mathbf{t}(\gamma) = 0$.

(2) Since $\gamma = \delta\beta$ and δ is nilpotent of degree 2:

$$div(V) \to \delta\gamma = \delta\delta\beta = 0$$

(3) Due to the Dirichlet boundary condition of harmonic gradients, $\mathbf{t}(\gamma) = 0$, the primitive φ will be constant on ∂M. This idea is comparable to a level set in the vector calculus context.

Moreover, we will discuss about the abundance of harmonic gradients

Let M have s connected components:

$$M = \sum_{i=1}^{s} M_i$$

and each boundary of the i-th component has r_i connected components:

$$\partial M_i = \sum_{j}^{r_i} \partial M_{ij}$$

So ∂M_{ij} is the j-th connected component of the boundary of the i-th connected component of M:

$$\partial M = \sum_{i=1}^{s} \sum_{j=1}^{r_i} \partial M_{ij}$$

In other words, ∂M has $m = r_1 + r_2 + \ldots + r_s$ components.

Lemma 4.4. [CDTG, Lemma 3]

$$\mathcal{H}_D^1(M) \simeq \mathbb{R}^{m-s}.$$

The previous lemma guarantees the abundance of harmonic gradients on M, because it supplies a concrete relation between the space of harmonic gradients and the topology of the domain, specifically, that the subspace of harmonic gradients HG is isomorphic to the homology group H_2. Therefore HG $\simeq \mathbb{R}^{m-s}$.

Lemma 4.5. [E, Proposition 4.12] *Let M be a compact domain in $\mathbb{R}^3$. Then*

$$H^2(M) \simeq H^1(M, \partial) \simeq \mathbb{R}^{m-s}.$$

Since $\mathcal{H}_D^1(M)$ is isomorphic to $\mathbf{H}_r^1(M) \simeq H^1(M, \partial M)$, the lemma above yields the following.

Lemma 4.6. [CDTG, Lemma 3]

$$\mathcal{H}_D^1(M) \simeq \mathbb{R}^{m-s}.$$

The previous lemma guarantees the abundance of harmonic gradients on M, because it supplies a concrete relation between the space of harmonic gradients and the topology of the domain, specifically, that the subspace of harmonic gradients HG is isomorphic to the homology group H_2. Therefore HG $\simeq \mathbb{R}^{m-s}$.

Example 4.7. Let M be the region between two concentric round spheres centered at the origin.

Once again, we are prone to consider the following question: how do we generate explicitly harmonic gradients from the knowledge of the numbers m and s of a compact domain Ω? The method, developed in [CDTG] reads as follows.

The desire to find vector fields which are the gradient of a function, divergence free and whose potential is locally constant on ∂M, translates to find a solution of the Laplace equation, $\Delta\varphi = 0$, on every connected component of M with their respective Dirichlet boundary conditions $\varphi|_{\partial M_{ij}} = c_{ij}$, where c_{ij} can be any constant. In this example, $m = 2$ and $s = 1$ so, in order to find a generator of HG, we will need to solve one Laplace equation with boundary conditions: $\varphi_{\partial M_1} = c_1$ and $\varphi_{\partial M_2 = c_2}$.

4.5. **Identifying GG and $d\Omega^0$.** In the context of vector calculus, the subspace of grounded gradients, GG, is defined as:

$$\text{GG} = \{V \in \text{VF}(M) \,|\, V = \nabla\varphi, \text{ for some } \varphi \in C^\infty(M),\ \varphi|_{\partial M} = 0\}$$

And in the language of differential forms, according to the decomposition proposed in the theorem 2.15, the grounded gradients could also be defined as:

$$\text{GG} = d\Omega^0(M) = \left\{df \mid f \in \Omega^0 \text{ and } \mathbf{t}(f) = 0\right\}$$

This interpretation is straightforward. Since $f \in \Omega^0$, the Dirichlet boundary condition is equivalent to say that the primitive, f, vanishes on ∂M.

5. Applications

In the vector calculus context, electromagnetic quantities can be divided in two groups, vector fields and scalar functions. However, the vector notation does not explicitly differentiate vector fields such as the magnetic field intensity, H, and the magnetic flux density, B, in the sense that, despite being numerically related, they represent different phenomena. H is a field intensity which is meant to be integrated over curves and B is a magnetic flux density which is meant to be integrated over surfaces.

Following the criteria before exposed, we can express some electromagnetic quantities in differential forms by looking at the dimension of the domain over which they are integrated. So in $\mathbb{R}^3$ we have:

1-forms: electric field E, magnetic field H

2-forms: electric flux density D, magnetic flux density B, electric current density J

3-forms: electric charge density ρ,

It is important to notice that we make these identifications because we know the physical properties of those vector fields and functions. However, we could have identified all the vector fields as 1-forms, as we made for the Hodge decomposition, and the Hodge star operator would serve as a bridge between both interpretation.

In the following section, we will describe Maxwell's equations in $\mathbb{R}^4$ in the language of differential forms and, for this purpose, we will identify all vector fields in $\mathbb{R}^3$ as 1-forms.

5.1. Maxwell's equations in differential forms.

One of the most remarkable advances in physics is the Maxwell's formulation of the equations of electromagnetism. We will see these equations can be described in a simpler notation when working with differential forms. Moreover, we will work out the proof in order to show that the exterior notation can not only simplify calculations, but also give further geometrical notions of the phenomenon. In fact, we will work out one the the various example proposed by Sjamaar in [Sj], specifically, the example 2.23.

In spacetime $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4), where $x_4 = ct$, Maxwell's equations can be written as follows:

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times H = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial D}{\partial t} \qquad \text{(Ampere's Law)}$$

$$\nabla \cdot D = 4\pi\rho \qquad \text{(Gauss' Law)}$$

$$\nabla \cdot B = 0 \qquad \text{(no magnetic monopoles)}$$

Where c is the speed of light, E is the electric field, H is the magnetic field, J is the density of electric current, ρ is the density of electric charge, B is the magnetic induction and D is the dielectric displacement. All of them depend on time, and also they all are vector fields, except ρ, which is a function on $\mathbb{R}^3$. Let's define the forms:

$$\alpha = (E_1dx_1 + E_2dx_2 + E_3dx_3)dx_4 + B_1dx_2dx_3 + B_2dx_3dx_1 + B_3dx_1dx_2$$

$$\beta = -(H_1dx_1 + H_2dx_2 + H_3dx_3)dx_4 + D_1dx_2dx_3 + D_2dx_3dx_1 + D_3dx_1dx_2$$

$$\gamma = \tfrac{1}{c}(J_1dx_2dx_3 + J_2dx_3dx_1 + J_3dx_1dx_2)dx_4 - \rho dx_1dx_2dx_3$$

5.1.1. *Maxwell's equations stated in differential forms.*

We will show now that Maxwell's equations are equivalent to:

$$d\alpha = 0$$

$$d\beta + 4\pi\gamma = 0$$

(1) It is easy to see that

$$d\alpha = (\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3})dx_2dx_3dx_4 + (\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1})dx_3dx_1dx_4 + (\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2})dx_1dx_2dx_4 +$$
$$(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3})dx_1dx_2dx_3 + \frac{\partial B_1}{\partial x_4}dx_2dx_3dx_4 + \frac{\partial B_2}{\partial x_4}dx_3dx_1dx_4 + \frac{\partial B_3}{\partial x_4}dx_1dx_2dx_4$$

In order to simplify the notations, we can instead work with $\star d\alpha$

$$\star d\alpha = (\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3})dx_1 + (\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1})dx_2 + (\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2})dx_3 +$$
$$(\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3})dx_4 + \frac{\partial B_1}{\partial x_4}dx_1 + \frac{\partial B_2}{\partial x_4}dx_2 + \frac{\partial B_3}{\partial x_4}dx_3$$

The Hodge star operator doesn't alter the fact that $d\alpha = 0$. By this mean, $d\alpha = 0$ implies $\star d\alpha = 0$, therefore, every coefficient must be zero. We have as a consequence:

$$\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} = -\frac{\partial B_1}{\partial x_4}$$
$$\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} = -\frac{\partial B_2}{\partial x_4}$$
$$\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} = -\frac{\partial B_3}{\partial x_4}$$
$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0.$$

Which we can write in vector notation as:

$$\nabla \times E = -\frac{\partial B}{\partial x_4}$$

We use now $x_4 = ct$, resulting in Faraday's Law:

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}.$$

Additionally, we can see that $div(B) = 0$ (no magnetic monopoles) by

$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = \nabla \cdot B = 0.$$

(2) We now work with the second equation $d\beta + 4\pi\gamma = 0$. Similarly, we calculate first $d\beta$:

$$d\beta = -(\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3})dx_2dx_3dx_4 - (\frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1})dx_3dx_1dx_4 - (\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2})dx_1dx_2dx_4 +$$
$$(\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3})dx_1dx_2dx_3 + \frac{\partial D_1}{\partial x_4}dx_2dx_3dx_4 + \frac{\partial D_2}{\partial x_4}dx_3dx_1dx_4 + + \frac{\partial D_3}{\partial x_4}dx_1dx_2dx_4$$

Then, we calculate $4\pi\gamma$:

$$4\pi\gamma = \frac{4\pi}{c}(J_1dx_2dx_3 + J_2dx_3dx_1 + J_3dx_1dx_2)dx_4 - 4\pi\rho dx_1dx_2dx_3$$

As we did previously, we will work out the demonstration using the Hodge star operator:

$$\star(d\beta + 4\pi\gamma) = \star d\beta + 4\pi \star \gamma = 0$$

In this case, we have:

$$\star d\beta = -(\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3})dx_1 - (\frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1})dx_2 - (\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2})dx_3 +$$

$$(\frac{\partial D_1}{\partial x_1}+\frac{\partial D_2}{\partial x_2}+\frac{\partial D_3}{\partial x_3})dx_4+\frac{\partial D_1}{\partial x_4}dx_1+\frac{\partial D_2}{\partial x_4}dx_2++\frac{\partial D_3}{\partial x_4}dx_3$$

$$4\pi\star\gamma=\frac{4\pi}{c}(J_1dx_1+J_2dx_2+J_3dx_3)-4\pi\rho dx_4$$

Therefore, if we want $\star d\beta+4\pi\star\gamma$ to be zero, every coefficient must be zero. Hence,

$$\frac{\partial H_3}{\partial x_2}-\frac{\partial H_2}{\partial x_3}=\frac{\partial D_1}{\partial x_4}+\frac{4\pi}{c}J_1$$

$$\frac{\partial H_1}{\partial x_3}-\frac{\partial H_3}{\partial x_1}=\frac{\partial D_2}{\partial x_4}+\frac{4\pi}{c}J_2$$

$$\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial x_2}=\frac{\partial D_3}{\partial x_4}+\frac{4\pi}{c}J_3$$

$$\frac{\partial D_1}{\partial x_1}+\frac{\partial D_2}{\partial x_2}+\frac{\partial D_3}{\partial x_3}=4\pi\rho.$$

In vector notation,

$$\nabla\times H=\frac{\partial D}{\partial x_4}+\frac{4\pi}{c}J$$

$$\nabla\times H=\frac{1}{c}\frac{\partial D}{\partial t}+\frac{4\pi}{c}J,$$

and

$$\nabla\cdot D=4\pi\rho$$

5.1.2. We will now show that γ is closed and that:

$$\nabla\cdot J+\frac{\partial\rho}{\partial t}=0$$

In order to show γ is closed, we are going to work out the exterior derivative.

$$\gamma=\frac{1}{c}(J_1dx_2dx_3+J_2dx_3dx_1+J_3dx_1dx_2)dx_4-\rho dx_1dx_2dx_3$$

$$d\gamma=\frac{1}{c}(\frac{\partial J_1}{\partial x_1}+\frac{\partial J_2}{\partial x_2}+\frac{\partial J_3}{\partial x_3})dx_1dx_2dx_3dx_4+\frac{\partial\rho}{\partial x_4}dx_1dx_2dx_3dx_4$$

We now use Ampere's Law in the next way:

$$\nabla\times H=\frac{4\pi}{c}J+\frac{1}{c}\frac{\partial D}{\partial t}$$

$$\nabla\cdot(\nabla\times H)=\nabla\cdot(\frac{4\pi}{c}J)+\nabla\cdot(\frac{1}{c}\frac{\partial D}{\partial t})$$

$$0=\frac{4\pi}{c}(\nabla\cdot J)+\frac{1}{c}(\nabla\cdot\frac{\partial D}{\partial t})$$

$$0=\frac{4\pi}{c}(\nabla\cdot J)+\frac{1}{c}\frac{\partial(\nabla\cdot D)}{\partial t}$$

And by Gauss' Law, we have $\nabla\cdot D=4\pi\rho$.

$$0=\frac{4\pi}{c}(\nabla\cdot J)+\frac{1}{c}\frac{\partial(4\pi\rho)}{\partial t}=\frac{4\pi}{c}(\nabla\cdot J)+\frac{4\pi}{c}\frac{\partial\rho}{\partial t}$$

Hence,

$$\nabla\cdot J+\frac{\partial\rho}{\partial t}=0$$

Which makes $d\gamma=0$, as we show next:

$$d\gamma=\frac{1}{c}(\frac{\partial J_1}{\partial x_1}+\frac{\partial J_2}{\partial x_2}+\frac{\partial J_3}{\partial x_3}+c\frac{\partial\rho}{\partial x_4})dx_1dx_2dx_3dx_4$$

where we use once again $x_4 = ct$

$$d\gamma = \frac{1}{c}(\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial \rho}{\partial t})dx_1dx_2dx_3dx_4 = \frac{1}{c}(\nabla \cdot J + \frac{\partial \rho}{\partial t})dx_1dx_2dx_3dx_4 = 0$$

$$\therefore d\gamma = 0$$

It is, γ is closed.

5.1.3. *Interpretation of α, β and γ.* Recall that we have defined three differential forms in order to write Maxwell's equations in terms of them, but we have not yet explained their meaning.

$$\alpha = (E_1dx_1 + E_2dx_2 + E_3dx_3)dx_4 + B_1dx_2dx_3 + B_2dx_3dx_1 + B_3dx_1dx_2$$

$$\beta = -(H_1dx_1 + H_2dx_2 + H_3dx_3)dx_4 + D_1dx_2dx_3 + D_2dx_3dx_1 + D_3dx_1dx_2$$

$$\gamma = \tfrac{1}{c}(J_1dx_2dx_3 + J_2dx_3dx_1 + J_3dx_1dx_2)dx_4 - \rho dx_1dx_2dx_3$$

We now define the following differential forms that are dual spaces of the vectors E, B, H and D.

$$\tilde{E} = E_1dx_1dx_4 + E_2dx_2dx_4 + E_3dx_3dx_4$$

$$\tilde{B} = B_1dx_1dx_4 + B_2dx_2dx_4 + B_3dx_3dx_4$$

$$\tilde{H} = H_1dx_1dx_4 + H_2dx_2dx_4 + H_3dx_3dx_4$$

$$\tilde{D} = D_1dx_1dx_4 + D_2dx_2dx_4 + D_3dx_3dx_4$$

$$\tilde{J} = J_1dx_1dx_4 + J_2dx_2dx_4 + J_3dx_3dx_4$$

And ρ is simply a 0-form. As we can see, all of them except γ seem to be 1-forms in $\mathbb{R}^3$ (they are actually in $\mathbb{R}^4$) that incorporate, via wedge product, the new base dx_4 that appears on the time-dependent case. It is, we think of the fields as fields living on spacetime. In terms of the these differential forms, we can rewrite α, β and γ as:

$$\alpha = \tilde{E} + \star\tilde{B}$$

$$\beta = -\tilde{H} + \star\tilde{D}$$

$$\gamma = \frac{1}{c}(\star\tilde{J})dx_4 - \rho dx_1dx_2dx_3$$

Note that α combines both the electric field and the magnetic induction in such a way that they are mutually orthogonal, since the Hodge star operator changes the basis to a perpendicular one on $\tilde{B}$. α is named the electromagnetic field, because it unifies both phenomenons and shows the fact that they are orthogonal. Similarly, β combines the effect of the magnetic field H and the dielectric displacement D to form a unified entity in the same way the electromagnetic field did. Its is reasonable since under certain conditions we can write H and D as linearly dependent of B and E, respectively.

On the other hand, γ is the result of combining the density of electric current J and the density of eletric charge ρ. We call γ the current.

5.1.4. *Showing that $\beta = \star_r\alpha$ in vacuum.* In order to work in Minkowski spacetime on $\mathbb{R}^{3+1}$, we define the relativistic Hodge star. If $\alpha = \sum_I f_I dx_I$,

$$\star_r\alpha = \sum_I f_I(\star_r dx_I)$$

where

$$\star_r dx_I = \begin{cases} \star dx_I & \text{if} \quad I \text{ contains } n+1 \\ -\star dx_I & \text{if} \quad I \text{ does not contain } n+1 \end{cases}$$

The Hodge star operator works like this in $\mathbb{R}^{3+1}$ because the inner product is redefined with a different metric in order to measure distances in spacetime, for more on this point of view, see [BM]. In this case, the fact that I contains $n+1$ means that dx_I contains dx_4. Hence, we can calculate the relativistic Hodge star of α:

$$\star_r \alpha = \star_r(\tilde{E} + \star\tilde{B}) = \star\tilde{E} - \tilde{B}$$

In vacuum, one has:

$$E = D$$
$$H = B.$$

Therefore, using these relations, β can be written as:

$$\beta = -\tilde{B} + \star\tilde{E},$$

and finally,

$$\beta = \star_r \alpha.$$

5.1.5. *Maxwell's equations on free space.* We define the free space as the vacuum without charges or currents. We now state its properties:

$$E = D,$$
$$H = B,$$
$$J = 0,$$
$$\rho = 0.$$

We had demonstrated that Maxwell's equations are equivalent to

$$d\alpha = 0,$$
$$d\beta + 4\pi\gamma = 0.$$

In free space, we have:

$$\gamma = \frac{1}{c}(\star\tilde{J})dx_4 - \rho dx_1 dx_2 dx_3 = 0$$

As J and ρ are zero. Consequently, Maxwell's equations are reduced to:

$$d\alpha = 0$$
$$d\beta = 0$$

We use the relation $\beta = \star_r \alpha$ and finally get that Maxwell's equations are equivalent to:

$$d\alpha = d \star_r \alpha = 0$$

5.2. **Graphical representation of differential forms.** As we have previously discussed, in comparison with vector analysis, the differential form representation seems to be more natural for integration. In the sense that there is not necessity of dot product because differential forms, thanks to the pullback, already carry that information.

Due to this intrinsic relationship between differential forms and integration, it is possible and convenient to graphically represent differential forms from the integral notion. This representation does not intend to be formal but a supporting intuitive feature for the understanding of differential forms. Here, we will present the representation of 1-forms, 2-forms and 3-forms in $\mathbb{R}^3$ and briefly explain why this representations are particularly useful for the study of the electromagnetic phenomenon.

Using the Hodge star operator, a 1-form can be represented as multiple surfaces distributed in the space, all of them perpendicular to the given 1-form (due to the geometrical properties of the Hodge star), and the integral of the 1-form over a path would be equal to the number of surfaces pierced by the path (considering rational numbers if the situation requires it). 2-forms are quantities that are integrated over surfaces and usually encode the notion of flux or fluid flow, in fact, due to the Hodge star again, a 2-form can be represented as an infinitesimal area together with an orthogonal direction to this infinitesimal area. Therefore, 2-forms are represented as a bundle of tubular sections that carry a unit amount of flux or fluid flow, so the integral of a 2-form over a surface is equal to the number of tubular sections that pierced the surface. Finally, 3-forms are meant to be integrated over a volume and usually encode the idea of density, indeed, as we saw in the previous sections, a 3-form is the infinitesimal of a volume form in $\mathbb{R}^3$. So, they are represented as small cubes which in the case of charge density carry a unit amount of electric charge. Note that this latter assertion is also justified by the Hodge star operator, because $\star dx_1 dx_2 dx_3 = 1$. For more on this point of view and some illustrative sketches, see [WR][SWA].

The differential form representation is attractive because, unlike the traditional representation of vector fields, it distinguishes physical quantities that are meant to be integrated over surfaces and the ones that are meant to be integrated over curves. For example, in the vector context, the magnetic field intensity (H), and the magnetic flux density (B) both are represented as a set of arrows distributed in the space, since they are both vector fields, which can lead to misinterpretations. With differential forms, by contrast, they are represented differently, as H is a 1-form (a work or circulation vector) and B is a 2-form (a flux vector).

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