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GRAPHS AND EQUIVARIANT COHOMOLOGY

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ABSTRACT. Let X be a T -skeletal variety, that is, a complex algebraic variety where a complex torus T acts with only finitely many fixed points and invariant curves. By a result of Goresky, Kottwitz and MacPherson, the equivariant cohomology of X can be read off from the associated graph of fixed points and invariant curves. The purpose of this paper is to compute explicitly and combinatorially the equivariant cohomology of certain projective toric surfaces and projective homogeneous spaces. In all these cases the equivariant cohomology is known to be a free module over a polynomial ring, and we provide explicit combinatorial and geometric bases for such modules. Furthermore, we exhibit an efficient algorithm to obtain such bases from a suitable order relation on the associated graph.

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1. INTRODUCTION

Let T be a complex algebraic torus. Let X be a T -skeletal variety, that is, a complex algebraic variety where T acts with only finitely many fixed points and invariant curves. Examples include flag varieties, toric varieties, and more generally, equivariant embeddings of reductive groups [Ti], [G2]. Goresky, Kottwitz and MacPherson [GKM] developed a theory, called *GKM*

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theory, that makes it possible to describe the equivariant cohomology of X , denoted $H_T^*(X)$, assuming certain technical conditions are met. Cohomology here is considered with rational coefficients. GKM theory states the following: if X is a complete T -skeletal variety whose cohomology vanishes in odd degrees, then the pullback to fixed points yields an injection $i^* : H_T^*(X) \rightarrow H_T^*(X^T)$, and the image can be described in terms of the combinatorial data coming from the fixed points and T -invariant curves. Consequently, $H_T^*(X)$ is identified with a ring of piecewise polynomial functions $PP_T^*(X)$ on the associated GKM graph Γ , i.e. the graph of fixed points and invariant curves (Theorem 2.23). Moreover, in this case, regardless of whether or not X is smooth, polynomial ring $PP_T^*(X)$ is a free module over the equivariant cohomology of a point (Lemma 2.26). Henceforth extensive research has been undertaken to obtain explicit combinatorial bases for these modules that take into account both, the geometry of the objects under study and the combinatorics of the GKM graph. For an overview of the research done in this area, as well as many explicit examples that highlight the combinatorial/algorithmical aspects of the theory, see the papers of Tymoczko [Ty], Brion [Br1], and Guillemin and Zara [GZ1], [GZ2].

GKM theory was initially applied, for the most part, to the cases when X is smooth and projective. Then, by a result of Bialynicki-Birula [B1], X admits a filtration by (disjoint, locally closed) affine spaces: the cells of the fixed points. This yields a basis of the equivariant cohomology ring of X as a free module over the equivariant cohomology ring of a point. Moreover, the GKM data coming from the invariant points and curves has been explicitly determined for interesting classes of smooth projective T -skeletal varieties: flag varieties (Carrell [Ti]), toric varieties (Danilov [D]) and equivariant embeddings of reductive groups (Brion [Br1] and Uma [U]). Later on, GKM theory was extended to incorporate a large class of possibly singular varieties, namely, \mathbb{Q} -filtrable varieties, by work of Gonzales [G1]. This class includes rationally smooth projective embeddings of reductive groups, and their equivariant cohomology and corresponding GKM data has been computed in [G2].

Explicit computations of equivariant cohomology rings in terms of GKM graphs are of particular interest not only to mathematicians but also to computer scientists and engineers working at the crossroads of discrete mathematics and geometric applications. See for instance [GZ1], [GZ2], the work of Sturmfels [S] and the references therein.

The purpose of this project is compute explicitly and combinatorially the equivariant cohomology $H_T^*(X)$, in the following cases:

- (1) When X is a projective smooth toric surface, in particular, for the projective plane and the Hirzebruch surface. See Examples 3.1 and 3.4.

- (2) When X is projective 3-space. See Examples 3.2 and 3.3. In Example 3.3 we consider \mathbb{P}^3 as the wonderful compactification of PGL_2 , and use the induced torus action to describe the cohomology.
- (3) When X is flag variety G/B or a partial flag variety G/P . In particular, we deal here with the case of GL_3/B and $Gr(2, 4)$ (the Grassmannian of planes through the origin in \mathbb{C}^4). See Examples 3.5 and 3.6.

In all cases (1) to (3), the equivariant cohomology has been described in very general terms in the references mentioned previously. Nevertheless, our goal is to provide explicit and combinatorial descriptions of these rings. These calculations will be relevant to people working in algebraic geometry and combinatorics. Finally, we develop an efficient algorithm to find such bases from a suitable order relation on the associated graph. This follows closely an algorithm stated in [Ty]. The algorithm and pseudo-code developed here should be of interest to people in Computer Science. We remark that for varieties where the the BB-decomposition (see Section 2.4) is not stratifiable (e.g. Example 3.4), then the present algorithm needs to be improved. We plan to do so by using the associated filtration and the notion of local indices developed in [G1]. This shall allow us to deal with singular varieties as well. The results will appear elsewhere.

This paper is organized as follows. In Section 2 we review the main notions and results from torus action on algebraic varieties needed for our study, in particular those related to cellular decompositions and GKM theory. In Section 3 we apply the theory from Section 2 to some concrete examples and describe in explicit combinatorial terms the equivariant cohomology (or graph cohomology) of an important class of T -skeletal varieties (Definition 2.22). Finally, in Section 4 we state the algorithm and pseudo-code produced and motivated by the explicit computations done in Section 3.

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2. PRELIMINARIES

2.1. Group actions. Our reference for the material in this section is [L]. Let G be a group. Broadly speaking, an action of G on a set X is a mechanism that allows group elements to move around the points of X (that is, G acts on X through symmetries). A precise definition is given next.

Definition 2.1. Let G be a group. Let X be a set. An **action** of G on X is a map

$$\mu : G \times X \rightarrow X, \quad (g, x) \mapsto g \cdot x$$

that satisfies the following properties:

- $e \cdot x = x$, for all $x \in X$, where $e \in G$ is the identity element of G .
- $(g \cdot h) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G$, and all $x \in X$.

A set X with an action of G is also called a G -set.

Definition 2.2. Let G be a group acting on a set X . Let x be a point of X . The **orbit** of x is the subset of X defined by

$$O(x) = \{y \in X \mid y = g \cdot x \text{ for some } g \in G\}$$

Equivalently,

$$O(x) = \{g \cdot x \mid g \in G\}.$$

In other words, the orbit of x consists of all the elements in X that can be reached from x under the action of G .

Note that when a group G acts on a set X , each point x of X has an orbit, and this orbit is a subset of X . Furthermore, the action of G on X yields a partition of X into disjoint orbits, where each orbit consists of the points that can be moved to each other by the group action.

Definition 2.3. Let X be a G -set. Consider a point $x \in X$. The **stabilizer** of x is the subgroup of G defined by

$$\text{Stab}(x) = \{g \in G \mid g \cdot x = x\}.$$

In other words, a group element g is in $\text{Stab}(x)$ if it does not move the point x .

Remark 2.4. Let X be a G -set. If $x \in X$, then the map $\varphi : G \rightarrow O(x)$, $g \mapsto g \cdot x$, induces a bijection

$$\bar{\varphi} : G/\text{Stab}(x) \rightarrow O(x),$$

by letting $\bar{\varphi}(g \cdot \text{Stab}(x)) = g \cdot x$. Hence, the coset space $G/\text{Stab}(x)$ and the orbit $O(x)$ are isomorphic (as G -sets). We write

$$G/\text{Stab}(x) \simeq O(x)$$

to refer to this isomorphism or bijection of sets. See [L] for more details.

In particular, if the action of G on X is transitive (i.e. if there is only one orbit), then X is isomorphic to $G/\text{Stab}(x)$, for some $x \in X$.

Definition 2.5. Let X be a G -set. A point $x \in X$ is called a **fixed point** if $g \cdot x = x$, for all $g \in G$. Equivalently, $x \in X$ is a fixed point if $\text{Stab}(x) = G$.

In plain words, x is a fixed point of the action if it cannot be moved around by any element of G . The **set of fixed points** is denoted by X^G .

Definition 2.6. Let X be a G -set. Let Y be a subset of X . We say that Y is **G -invariant** if for all $g \in G$ and all $y \in Y$, we have $g \cdot y \in Y$.

Informally speaking, a subset Y is G -invariant if the group elements move around the points of Y without making them land outside of Y .

It follows from the definition that a G -invariant subset is also a G -set (because it admits an induced action of G).

Definition 2.7. Let X be a G -set. We denote by X/G the set of orbits of the action, and call X/G the **orbit space** or **quotient space**. That is,

$$X/G = \{O(x) \mid x \in X\}$$

For simplicity, the points of X/G are often written as $[x]$, so that $[x] = [y]$ if and only if $O(x) = O(y)$. Hence, as mentioned before,

$$X = \bigsqcup_{[x] \in X/G} O(x) \quad (\text{disjoint union}).$$

Example 2.8. Let G be the multiplicative group $\mathbb{C} - \{0\}$ (this group is also denoted by \mathbb{C}^*). Let V denote the vector space \mathbb{C}^{n+1} . Consider the action of G on V given by

$$t \cdot (x_0, x_1, \dots, x_n) = (tx_0, tx_1, \dots, tx_n),$$

where $t \in G$ and $x = (x_0, \dots, x_n) \in V$.

It is easily checked that $0 \in V$ is the only fixed point of the action. On the other hand, if $x \in V$ and $x \neq 0$, then $\text{Stab}(x) = \{1\}$. Moreover, if $x \in V$ and $x \neq 0$, then the orbit of x consists of all non-zero multiples of x . Put another way, if L_x denotes the line in \mathbb{C}^{n+1} that passes through 0 and x , then $O(x) = L_x - \{0\}$.

Now let $X = V - \{0\}$ (with the induced action of G). Consider the orbit space X/G . This orbit space is called **the complex projective space** \mathbb{P}^n . By construction, \mathbb{P}^n parametrizes the set of lines that pass through the origin in \mathbb{C}^{n+1} . A point of \mathbb{P}^n is denoted by its *homogeneous coordinates* $[x_0 : x_1 : \dots : x_n]$. Thus $[x_0 : x_1 : \dots : x_n]$ and $[y_0 : y_1 : \dots : y_n]$ represent the same point of \mathbb{P}^n if and only if $(y_0, y_1, \dots, y_n) = (\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ for some $\lambda \in \mathbb{C}^*$. For instance, $[1 : 1 : 2] = [2 : 2 : 4]$ and $[-1 : 0 : 1] = [i : 0 : -i]$ in \mathbb{P}^2 . Likewise, $[0 : 0 : \pi] = [0 : 0 : 1]$ in \mathbb{P}^2 . Furthermore, by construction, $[x_0 : x_1 : \dots : x_n]$ are homogeneous coordinates of a point in \mathbb{P}^n if and only if at least one of the coordinates x_i is nonzero (equivalently, the x_i 's cannot be simultaneously zero). Hence, the homogeneous coordinates in \mathbb{P}^n emphasize the fact that a line through the origin in \mathbb{C}^{n+1} is uniquely determined by its direction.

2.2. Complex Projective Space. We mention here some important properties of projective space. Let \mathbb{P}^n be (complex) projective space, with homogeneous coordinates $[x_0 : x_1 : x_2 : \dots : x_n]$. Recall that the coordinates x_i cannot be simultaneously zero. For each $i = 0, 1, \dots, n$, let

$$U_i = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$$

Clearly,

$$U_i = \{[x_0 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : a_n] \in \mathbb{P}^n\}.$$

In other words, each U_i is isomorphic to a copy of \mathbb{C}^n embedded in \mathbb{P}^n . It follows that we have a natural covering of \mathbb{P}^n by affine patches or (affine charts):

$$\mathbb{P}^n = \bigcup_{i=0}^n U_i.$$

Hence, we can view \mathbb{P}^n as a global object built up from the union of $n + 1$ (overlapping) affine patches, each corresponding to a copy of \mathbb{C}^n embedded in \mathbb{P}^n . Note that every point of \mathbb{P}^n lies in at least one affine patch.

For instance, let us describe the situation in the particular case of \mathbb{P}^2 . Let $P = [x : y : z]$ be a point in \mathbb{P}^2 .

- If $x \neq 0$, then $[x : y : z] = [1 : \frac{y}{x} : \frac{z}{x}]$, since both coordinates correspond to the same direction. Therefore, $(\frac{y}{x}, \frac{z}{x}) \in \mathbb{C}^2$.
- If $y \neq 0$, then $[x : y : z] = [\frac{x}{y} : 1 : \frac{z}{y}]$, since they yield the same direction. Therefore, $(\frac{x}{y}, \frac{z}{y}) \in \mathbb{C}^2$.
- If $z \neq 0$, then $[x : y : z]$ and $[\frac{x}{z} : \frac{y}{z} : 1]$ correspond to the same direction, that is, $[x : y : z] = [\frac{x}{z} : \frac{y}{z} : 1]$. Therefore, $(\frac{x}{z}, \frac{y}{z}) \in \mathbb{C}^2$.

Thus, \mathbb{P}^2 is the union of three copies of \mathbb{C}^2 . Moreover, observe that there are transition functions between these charts or local coordinates. Indeed, if a point $P \in \mathbb{P}^2$ lies on both U_x and U_y , then P has two seemingly different local representations: it can be written as $[1 : \frac{y}{x} : \frac{z}{x}]$ in U_x , whereas it can be written as $[\frac{x}{y} : 1 : \frac{z}{y}]$ in U_y . Nevertheless, these two viewpoints are related: to pass from the representation in U_x to the one in U_y we use the transition function *multiplication by $\frac{x}{y}$* . Clearly, we multiply by $\frac{y}{x}$ to pass from U_y to U_x . The general situation for arbitrary \mathbb{P}^n is quite similar. Just as a surface or, more generally, a manifold, we can view projective space as being locally defined in terms of affine patches and transition functions that allow to transfer information between them, whenever they overlap. See [H].

Back to the general case, for each i , let H_i be the complement of U_i , namely $H_i = \mathbb{P}^n - U_i = \{x_i = 0\}$. Note that each H_i corresponds to a copy of \mathbb{P}^{n-1} embedded in \mathbb{P}^n . The various H_i are called canonical hyperplanes in \mathbb{P}^n . If we pick a hyperplane, say H_0 , we can partition \mathbb{P}^n as

$$\mathbb{P}^n = U_0 \sqcup H_0 \simeq \mathbb{C}^n \sqcup \mathbb{P}^{n-1},$$

and repeating this process we get

$$\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \dots \sqcup \mathbb{C} \sqcup \{\text{pt}\},$$

where pt represents a single point in \mathbb{P}^n . Consequently, projective space \mathbb{P}^n has a cell decomposition and can be build inductively from \mathbb{P}^{n-1} . After introducing projective varieties in the next section we shall go back to this idea of decomposing a variety into affine cells.

2.3. Algebraic varieties. We recollect a few basic notions on algebraic varieties. For a complete treatment of the subject, and proofs of the results stated here, the reader is invited to consult [H, Chapter 1].

Definition 2.9. Let $\mathbb{C}[x_1, \dots, x_n]$ be the ring of polynomials in n variables over \mathbb{C} .

- (1) For a set $S \subset \mathbb{C}[x_1, \dots, x_n]$ of polynomials, the **affine variety** given by S is defined as

$$V(S) = \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in S\}.$$

- (2) A subset $X \subseteq \mathbb{C}^n$ is called an **affine variety** if X is the affine algebraic variety given by a set $S \subset \mathbb{C}[x_1, \dots, x_n]$ of polynomials.

Affine varieties are also called affine *algebraic* varieties. An affine variety $V(S)$ can be described as the common zeros of the polynomials in S . Intuitively, an affine variety is a subset of \mathbb{C}^n which is “cut out” from \mathbb{C}^n by some collection of polynomials.

When the set S consists of a single polynomial f , then we write $V(f)$ for the corresponding variety. In this case, the affine variety $V(f)$ is called a **hypersurface** in \mathbb{C}^n .

By Hilbert’s Basis Theorem, every affine variety can in fact be written as $V(I)$ for some *finite* collection I of polynomials. In other words, every affine variety in \mathbb{C}^n can be obtained by intersecting a finite number of hypersurfaces in \mathbb{C}^n .

Next we describe the class of objects most important for our purposes.

Definition 2.10. Let $\mathbb{C}[x_0, \dots, x_n]$ be the ring of polynomials in $n + 1$ variables over \mathbb{C} .

- (1) Let S be any set of *homogeneous* polynomials in $\mathbb{C}[x_0, x_1, \dots, x_n]$. The **projective variety** associated to S is defined as

$$V(S) = \{p \in \mathbb{P}^n \mid f(p) = 0 \text{ for all } f \in S\}.$$

- (2) A subset $X \subseteq \mathbb{P}^n$ is called a **projective variety** if X is the projective variety associated to some set $S \subset \mathbb{C}[x_0, x_1, \dots, x_n]$ of homogeneous polynomials.

Projective varieties are also called projective *algebraic* varieties. By the homogeneous version of Hilbert’s Basis Theorem, every projective variety can be written as $V(J)$ for some *finite* collection of homogeneous polynomials. Hence, as in the affine case, every projective variety in \mathbb{P}^n can be described as a finite intersection of homogeneous hypersurfaces in \mathbb{P}^n .

It can be shown that: (i) the union of two projective varieties is a projective variety, (ii) the intersection of any family of projective varieties is a projective variety, and (iii) the empty set and \mathbb{P}^n are projective varieties (indeed $\emptyset = V(1)$ and $\mathbb{P}^n = V(0)$). Thus, by declaring projective varieties as the **closed subsets** of \mathbb{P}^n , we get the Zariski topology of \mathbb{P}^n . In this context,

the closed subsets of \mathbb{P}^n are of the form $V(S)$, and the **open subsets** of \mathbb{P}^n are of the form $\mathbb{P}^n - V(S)$, for some subset S of homogeneous polynomials (i.e. the open subsets of \mathbb{P}^n are the complements of projective varieties). For more details, see [H].

Let Z be any subset of \mathbb{P}^n . The **closure** \overline{Z} of Z in \mathbb{P}^n is defined as the smallest projective variety in \mathbb{P}^n that contains Z . More concretely, $\overline{Z} = V(I_Z)$ where I_Z is the set of all homogeneous polynomials that vanish at Z .

By the discussion of Subsection 2.2, if $Y \subset \mathbb{C}^n$ is an affine variety, we can embed it in \mathbb{P}^n by identifying \mathbb{C}^n with the affine patch U_0 of \mathbb{P}^n . We can then take the *projective closure* \overline{Y} of Y in \mathbb{P}^n . Hence, we can consider an affine variety $Y \subset \mathbb{C}^n$ as being embedded in the projective variety $\overline{Y} \subset \mathbb{P}^n$, as an open subset. Furthermore, it can be shown that any projective variety can be written as a finite union of affine varieties, much in the same way as we described \mathbb{P}^n as a union of affine charts in Section 2.2. See [H].

In the context of algebraic varieties it is possible to define the notion of dimension of a projective variety $X = V(S)$. Informally speaking, this refers to the local number of variables needed to describe X . We refer to [H] for the appropriate definition. For us, it suffices to state the following facts:

- The (complex) dimension of \mathbb{C}^n and \mathbb{P}^n is n .
- The dimension of a hypersurface $V(f) \subset \mathbb{P}^n$ is $n - 1$. Similarly, the dimension of an affine hypersurface $V(g) \subset \mathbb{C}^n$ is $n - 1$.
- If G is a group, H a closed subgroup, and G/H is a variety, then the dimension of G/H is $\dim(G) - \dim H$. This shall apply in particular to the orbits of an algebraic group acting on a projective variety.

There is also the notion of **smooth variety**. Since for the most part we work with either hypersurfaces or homogeneous spaces (spaces where a group acts transitively), we only state the following facts.

- \mathbb{C}^n and \mathbb{P}^n are smooth varieties. In fact, any open subset of these varieties is smooth.
- A hypersurface $V(f) \subset \mathbb{P}^n$ is smooth if and only if the gradient of f does not vanish anywhere in $V(f)$, that is, $\nabla f(P) = (\frac{\partial f}{\partial x_0}(P), \dots, \frac{\partial f}{\partial x_n}(P)) \neq \vec{0}$, for all $P \in V(f)$. Similarly for smooth affine hypersurfaces.
- If G is a group, and H a closed subgroup, then G/H is a quasiprojective variety (i.e. an open subset of a projective variety). Moreover, G/H is smooth.

Next we provide some additional concrete examples relevant to our study.

Example 2.11. Let n be a nonnegative integer. The **Hirzebruch surface** $\mathbb{F}(n)$ is the subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ defined by

$$\mathbb{F}(n) = \{([x_0 : x_1 : x_2], [y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid x_1 y_1^n = x_2 y_2^n\}.$$

The Hirzebruch surface is a smooth projective variety. It is worth mentioning the following facts (see [H]):

- $\mathbb{F}(0) = \mathbb{P}^1 \times \mathbb{P}^1$.

- $\mathbb{F}(1) = \text{Bl}(\mathbb{P}^2)$, the blow-up of \mathbb{P}^2 at the point $[1 : 0 : 0]$.
- $\mathbb{F}(n)$ and $\mathbb{F}(m)$ are not isomorphic (as projective varieties) if $n \neq m$.

It is also a well-known fact that \mathbb{P}^2 is not isomorphic to any of the Hirzebruch surfaces (see e.g. Example 2.19).

The surface \mathbb{F}_n comes equipped with an important family of rational projective curves (i.e. curves isomorphic to \mathbb{P}^1). For later use, we shall describe them here. Let $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}^1$ be the map given by projection to the second factor. It is known that π_n is a \mathbb{P}^1 -bundle. Denote by L_0 and L_∞ the fibers over $[1 : 0]$ and $[0 : 1]$ respectively. On the other hand, consider the morphism $\varphi_n : \mathbb{F}_n \rightarrow \mathbb{P}^2$ given by the first projection. Distinguished curves are $C_- = \varphi_n^{-1}([1 : 0 : 0])$ and $C_+ = \varphi_n^{-1}([0 : x_1 : x_2])$. These four curves L_0, L_∞, C_+, C_- of \mathbb{F}_n shall reappear later on when we consider torus actions on varieties.

Example 2.12. Let G be GL_n , the group of invertible $n \times n$ matrices. Let \mathcal{F} be the set of *complete flags* in \mathbb{C}^n . That is, the elements of \mathcal{F} are nested chains

$$V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n,$$

of linear subspaces of \mathbb{C}^n , where $\dim(V_i) = i$, for $i = 0, 1, \dots, n$.

Note that G acts naturally on \mathcal{F} . Let $x \in \mathcal{F}$ be the canonical flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_n \rangle,$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{C}^n . One easily checks that the stabilizer of x is the subgroup $B \subset G$ that consists of the upper triangular matrices. Since this action is transitive, we get $\mathcal{F} \simeq G/B$ (as sets). It is known that this is an isomorphism of varieties (see e.g. [Hu]). The smooth projective variety \mathcal{F} is called the **full flag variety** of G . It has dimension $\frac{n(n-1)}{2}$.

Example 2.13. Consider \mathbb{P}^5 with homogeneous coordinates $[x_1 : x_2 : x_3 : y_1 : y_2 : y_3]$. Let g be the homogeneous polynomial

$$g(x_1, x_2, x_3, y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3$$

Then $V(g)$, the zero locus of g in \mathbb{P}^5 , is a projective smooth variety of dimension 4. It corresponds to the Plücker embedding of the Grassmannian $Gr(2, 4)$ in \mathbb{P}^5 . The Grassmannian $Gr(2, 4)$ parametrizes planes through the origin in \mathbb{C}^4 .

Projective spaces together with the algebraic varieties in Examples 2.11, 2.12, 2.13 form our class of standard examples in the sequel.

2.4. Torus actions on varieties. An *algebraic group* is an algebraic variety G equipped with the structure of a group, such that the group operations (multiplication and taking inverses) are morphisms of varieties (i.e. are defined locally by polynomials). Likewise, an action of an algebraic group G on a variety X is an action $\mu : G \times X \rightarrow X$ which is also a morphism of

varieties. In this context, a **G -variety** is a variety with an action of G . We refer to [Hu] for the appropriate details. In this paper, we shall work mostly with algebraic tori. An algebraic **torus** T is an algebraic group isomorphic to $(\mathbb{C}^*)^d$ (with the usual multiplication).

Let $T = (\mathbb{C}^*)^d$ be an algebraic torus. A **character** of T is a homomorphism (of algebraic groups) from T to \mathbb{C}^* . If $\chi : T \rightarrow \mathbb{C}^*$ is a character, then there are integers $a_1, \dots, a_d \in \mathbb{Z}$ such that the following holds

$$\chi(t_1, \dots, t_d) = t_1^{a_1} \cdots t_d^{a_d}.$$

See [Hu]. The d -tuple $(a_1, \dots, a_d) \in \mathbb{Z}^d$ is called the **weight** of the character χ , and each character is uniquely identified by its weight. Hence, we often refer to a character of T as a *weight of T* . The set of characters of T forms a group under pointwise multiplication. We write Δ for the group of characters of T .

A **one-parameter subgroup** of T is a homomorphism $\lambda : \mathbb{C}^* \rightarrow T$. Any one-parameter subgroup of T can be written uniquely as $\lambda(s) = (s^{b_1}, \dots, s^{b_d})$, for some integers $b_1, \dots, b_d \in \mathbb{Z}$. See [Hu]. The set of one-parameter subgroups of T forms a group under pointwise multiplication. This group is denoted by Δ^* .

Now let $\lambda \in \Delta^*$ and $\chi \in \Delta$. Then $\chi \circ \lambda$ is a character of \mathbb{C}^* . Hence, $(\chi \circ \lambda)(s) = s^r$, for some integer r . We shall denote this integer by $\langle \lambda, \chi \rangle$. One easily checks that

$$\langle \lambda, \chi \rangle = (a_1, \dots, a_d) \cdot (b_1, \dots, b_d) = a_1 b_1 + \dots + a_d b_d.$$

Consequently, we get a natural pairing between Δ^* and Δ defined by

$$\langle \cdot, \cdot \rangle : \Delta^* \times \Delta \rightarrow \mathbb{Z}, \quad (\lambda, \chi) \mapsto \langle \lambda, \chi \rangle.$$

Example 2.14. Let $T = (\mathbb{C}^*)^d$ be a torus. Let \mathbb{P}^n be projective space. It is a well-known fact from the theory of algebraic groups that any (algebraic) torus action on \mathbb{P}^n can be written as

$$t \cdot [x_0, x_1, \dots, x_n] = [\chi_0(t)x_0 : \chi_1(t)x_1, \dots, \chi_n(t)x_n],$$

for some weights $\chi_0, \chi_1, \dots, \chi_n$ of T . See [Hu]. See also Section 3 for more concrete examples in this vein.

Convention: Henceforth, we shall assume that the projective T -varieties under consideration admit a T -equivariant embedding into some projective space \mathbb{P}^N , where T acts on \mathbb{P}^N as in Example 2.14. This assumption is fulfilled by T -stable subvarieties of normal (e.g. smooth) projective T -varieties [Su].

2.4.1. *The Bialynicki-Birula decomposition.* The results in this subsection are due to Bialynicki-Birula [B1], [B2]. We focus on the case of torus actions with only a finite number of fixed points, since this suffices for our purposes.

Let T be an algebraic torus. Let X be a projective T -variety with only finitely many fixed points. We write $X^T = \{x_1, \dots, x_m\}$. A one-parameter

subgroup $\lambda : \mathbb{C}^* \rightarrow T$ is called **generic** if $X^{\mathbb{C}^*} = X^T$, where \mathbb{C}^* acts on X via λ . Generic one-parameter subgroups always exist. (See Subsection 2.4.2).

Now fix a generic one-parameter subgroup λ of T . For each i , define the subset

$$W_+(x_i, \lambda) = \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t) \cdot x = x_i\}.$$

Note that the cells $W_+(x_i, \lambda)$, obtained using λ , are T -invariant. Moreover, since X is projective, the (disjoint) union of these cells covers all of X , that is,

$$X = \bigsqcup_{i=1}^m W_+(x_i, \lambda).$$

The decomposition $\{W_+(x_i, \lambda)\}_{i=1}^m$ is called the (plus) Bialynicki-Birula decomposition, or **BB-decomposition**, of X associated to λ . Each $W_+(x_i, \lambda)$ is called a **cell** of the decomposition.

Similarly, one can define the *minus* decomposition of X , taking as corresponding cells

$$W_-(x_i, \lambda) = \{x \in X \mid \lim_{t \rightarrow \infty} \lambda(t) \cdot x = x_i\}.$$

Accordingly, one obtains a corresponding (minus) BB-decomposition of X .

Definition 2.15. Let X be a T -variety with X^T finite. Let $\{W_+(x_i, \lambda)\}_{i=1}^m$ be the BB-decomposition associated to some generic one-parameter subgroup λ of T . The decomposition $\{W_+(x_i, \lambda)\}$ is called **filtrable** if there exists a finite increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \dots \subset \Sigma_m$ of T -invariant closed subvarieties of X such that:

- a) $\Sigma_0 = \emptyset$, $\Sigma_m = X$,
- b) $\Sigma_j \setminus \Sigma_{j-1}$ is a cell of the decomposition $\{W_+(x_i, \lambda)\}$, for each $j = 1, \dots, m$.

We shall call Σ_j the *j -th filtered piece* of X . In this context, it is common to say that X is **filtrable**. If, moreover, the cells $W_+(x_i, \lambda)$ are isomorphic to affine spaces \mathbb{C}^{n_i} , then X is called **T -cellular**.

Clearly, one can also define the previous notions replacing the plus BB-decomposition with the minus BB-decomposition, as the reader can easily check.

Theorem 2.16 ([B1], [B2]). *Let X be a projective T -variety with only finitely many fixed points. Let λ be a generic one-parameter subgroup. Then the associated BB-decompositions $\{W_+(x_i, \lambda)\}$ and $\{W_-(x_i, \lambda)\}$ are filtrable. Furthermore, if X is smooth, then X is T -cellular.* \square

Let X be a projective T -variety with only a finite number of fixed points. If X is smooth, then Theorem 2.16 implies that X can be build inductively from affine spaces, by moving up in the filtration and attaching one cell at a time. This process is systematic and preserves cohomology in lower and higher degrees at each step. A more precise statement is given next. For a proof, see [B1].

Corollary 2.17. *Let X be a projective T -variety with only a finite number of fixed points. If X is T -cellular, then X has vanishing odd cohomology over the rationals, and the dimension of $H^{2k}(X)$, the cohomology group of X in even degree $2k$, equals the number of affine cells of complex dimension k . Furthermore, $\dim_{\mathbb{Q}} H^*(X) = |X^T|$, where $|X^T|$ denotes the number of fixed points. \square*

Recall that the dimension of the vector space $H^k(X)$ is also called the **k -th Betti number** of X , and it is denoted by $b_k = b_k(X)$. Hence, the previous result asserts that if X is a smooth projective T -variety with X^T finite, then X has a paving by affine cells, the odd Betti numbers b_{2k+1} are zero, and the even Betti numbers b_{2k} are obtained by counting the number of cells in X of the form \mathbb{C}^k .

2.4.2. Generic one-parameter subgroups. Let X be a projective T -variety. A one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ is called **generic** if $X^{\mathbb{C}^*} = X^T$, where \mathbb{C}^* acts on X via λ . We prove here that generic one-parameter subgroups always exist. We provide an argument only for the case of interest to us, namely for T -skeletal varieties, and refer to [Br1], [G3] for the general case. If X is T -skeletal, then there are finitely many characters χ_1, \dots, χ_k of T associated to the T -invariant curves of X . So we can choose a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ such that $\langle \lambda, \chi_i \rangle \neq 0$ for all $i = 1, \dots, k$. One easily checks that $X^{\mathbb{C}^*} = X^T$, that is, λ is generic.

A choice of generic one-parameter subgroup is equivalent to a choice a weight vector $\mathbf{w} = (w_1, \dots, w_d)$ (the weight of λ) such that \mathbf{w} is not orthogonal to the vector weights $\mathbf{v}_1, \dots, \mathbf{v}_k$ of the characters χ_1, \dots, χ_k of T . The weight vector \mathbf{w} of a generic one-parameter subgroup is often called a **polarizing** vector of the action.

Let X be a smooth projective T -variety with finitely many fixed points. Choosing a generic λ yields a paving of X by affine spaces or cells (Theorem 2.16 and Corollary 2.17).

2.4.3. Examples of BB-decompositions. We apply the results on torus actions and cellular decompositions developed in this subsection to four basic examples.

Example 2.18. A basic example is \mathbb{P}^n . Let $T = \mathbb{C}^*$, and choose integers $a_0 < a_1 < \dots < a_n$. Now define a T -action on \mathbb{P}^n via

$$s \cdot [x_0, x_1, \dots, x_n] = [s^{a_0} x_0 : s^{a_1} x_1 : \dots : s^{a_n} x_n].$$

The fixed points are the points $[e_0], [e_1], \dots, [e_n]$ of \mathbb{P}^n corresponding to the coordinate lines in \mathbb{C}^{n+1} , namely, $[e_i] = [0 : \dots : 0 : 1 : 0 : \dots : 0]$, where there is exactly one nonzero entry in the i -th position, for $i = 0, 1, \dots, n$. For a given i , the associated cell is

$$W_+(e_i) = \{[0 : 0 : \dots : 1 : x_{i+1}, \dots, x_n]\}$$

where $x_{i+1} : \dots : x_n$ are arbitrary complex numbers. Thus $W_+(e_i)$ is isomorphic to \mathbb{C}^{n-i} .

In accordance with Theorem 2.16, this cellular decomposition can be filtered as follows. For each i , let $X_i = \bigcup_{j \geq i} W_+(e_j)$. Then we get a nested sequence of closed T -invariant subvarieties

$$X_0 \supset X_1 \supset \dots \supset X_{n-1} \supset X_n = \{[e_n]\}.$$

Note that this induces a total ordering of the fixed points:

$$e_0 \geq e_1 \geq \dots \geq e_n$$

We can be even more explicit about the filtration described above. Indeed, each X_i is a projective space of smaller dimension. That is, the filtration above corresponds to the filtration

$$\mathbb{P}^n \supset \mathbb{P}^{n-1} \supset \dots \supset \mathbb{P}^1 \supset \{\text{pt}\},$$

already known from Subsection 2.2.

Likewise, the minus decomposition reads as follows. For a given i , we get

$$W_-(e_i) = \{[x_0 : x_1 : \dots : x_{i-1} : 1 : 0 : 0 \dots : 0]\}$$

where x_0, \dots, x_{i-1} are arbitrary complex numbers. Thus $W_-(e_i)$ is isomorphic to \mathbb{C}^i . This decomposition is filtrable. Indeed, let $Z_i = \bigcup_{j \leq i} W_-(e_j)$. Thus each Z_i is a closed T -stable subvariety of \mathbb{P}^n and we have

$$\{e_0\} = Z_0 \subset Z_1 \subset \dots \subset Z_{n-1} \subset Z_n = \mathbb{P}^n.$$

Once again, this corresponds to the canonical filtration of \mathbb{P}^n by lower dimensional projective spaces. In this case, the total ordering of the fixed points is the following:

$$e_0 \leq e_1 \leq \dots \leq e_n.$$

Finally, from Corollary 2.17 it follows that the odd Betti numbers of \mathbb{P}^n are zero, and the even Betti numbers of \mathbb{P}^n are equal to 1. The next diagram describes pictorially this cell decomposition for the case $n = 3$:

$$\begin{array}{c} \mathbb{C}^3 \\ | \\ \mathbb{C}^2 \\ | \\ \mathbb{C} \\ | \\ \text{pt} \end{array}$$

Example 2.19. Let \mathbb{F}_n be the Hirzebruch surface defined in Example 2.11. Consider the \mathbb{C}^* -action on $\mathbb{F}(n)$ given by

$$t \cdot ([x_0 : x_1 : x_2], [y_1 : y_2]) = ([t^n x_0 : t^{2n} x_1 : x_2], [y_1 : t^2 y_2]).$$

The fixed points are the points

$$x = ([0 : 0 : 1], [1 : 0]),$$

$$y = ([0 : 1 : 0], [0 : 1]),$$

$$z = ([1 : 0 : 0], [1 : 0]),$$

and

$$w = ([1 : 0 : 0], [0 : 1]).$$

The BB-cells associated to these fixed points are:

- $C_y = \{y\}$
- $C_w = \{([1 : x_1 : 0], [0 : 1]) \mid x_1 \in \mathbb{C}\} \simeq \mathbb{C}$
- $C_z = \{([1 : 0 : 0], [1 : y_2]) \mid y_2 \in \mathbb{C}\} \simeq \mathbb{C}$
- $C_x = \{([x_0 : y_2^n : 1], [1 : y_2]) \mid x_0, y_2 \in \mathbb{C}\} \simeq \mathbb{C}^2$

From Corollary 2.17 it follows that the odd Betti numbers of \mathbb{F}_n are zero, and the even Betti numbers of \mathbb{F}_n are $b_0 = 1$, $b_2 = 2$, $b_4 = 1$. Clearly, \mathbb{F}_n is not isomorphic to \mathbb{P}^2 , since they have different Betti numbers. Moreover, with the notation of Example 2.11, one easily checks that $C_- = \overline{C_z}$ and $L_\infty = \overline{C_w}$. It follows that L_∞ and C_- freely generate the (co)homology group $H^2(\mathbb{F}_n)$.

In this case, the associated filtration of \mathbb{F}_n by closed subvarieties is (Theorem 2.16):

$$Z_y \subseteq Z_w \subseteq Z_z \subseteq Z_x$$

where

$$Z_y = \{y\}$$

$$Z_w - Z_y = C_w$$

$$Z_z - Z_w = C_z$$

and

$$Z_x - Z_z = C_x.$$

Thus, using this action, we get a total order of the fixed points:

$$y \leq w \leq z \leq x.$$

Note that we can get the reverse ordering by considering the minus BB-decomposition, as we saw in the previous example.

Pictorially, we have

$$\begin{array}{ccc} \mathbb{C}_z - \mathbb{C}_w & & \\ | & & | \\ \mathbb{C}^2 - \{y\} & & \end{array}$$

In this case, the BB-decomposition is not a stratification (i.e. the closure of cells are not unions of cells). See [B2].

Example 2.20. Let $G = GL_n$. Denote by B the subgroup of upper triangular matrices, and by B^- the subgroup of lower triangular matrices. As we mentioned in Example 2.12, the quotient space G/B is a smooth projective variety, called the (full) flag variety of G . An elementary result from linear algebra asserts the following: given a matrix $A \in G$, there are matrices L , P and U , where P is a (unique) permutation matrix, L is a lower triangular matrix (i.e. $L \in B^-$) and U is an upper triangular matrix (i.e. $U \in B$), such that A can be written as $A = LPU$. In other words, any invertible matrix A can be reduced by a sequence of elementary row and column operations into a permutation matrix (recall that multiplying by L codifies elementary row operations performed only in a downwards fashion, whereas multiplying by U codifies column operations, where we move solely from left to right). Let w_0 be the permutation matrix

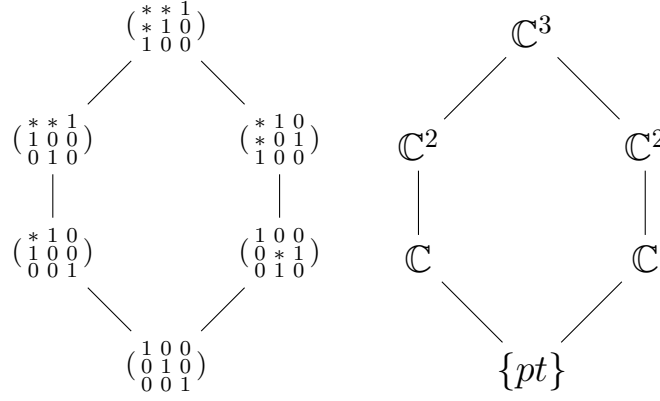
$$w_0 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & \cdot & \\ 1 & & & \end{pmatrix}.$$

Since $B^- = w_0 B w_0$, the result mentioned in the previous paragraph implies the **Bruhat decomposition** of an invertible matrix A . Namely, A can be written as a product $A = U_1 P U_2$, for appropriate upper triangular matrices U_1, U_2 , and a unique matrix permutation P . From here it follows that

$$G = \bigsqcup_{s \in S_n} B \cdot s \cdot B^-,$$

where S_n is the symmetric group or permutation group. (Note that B and B^- can be exchanged appropriately in the decomposition of G by using w_0 .) In particular, we obtained the associated **Bruhat decomposition of the flag variety** G/B , namely $G/B = \bigsqcup_{s \in S_n} B[s]$. It is well-known that the cells $B[s]$ are isomorphic to affine spaces $\mathbb{C}^{\ell(w)}$. See [Hu]. Moreover, the maximal torus T of G (corresponding to the diagonal matrices) acts on G/B with finitely many fixed points and T -invariant curves (i.e. G/B is T -skeletal, see next Subsection 2.5). By choosing an appropriate generic subgroup λ it is possible to show that the Bruhat cells $B[s]$ correspond to the plus decomposition of G/B induced by λ . See e.g. [Br1].

Next we display that Bruhat cells associated to GL/B for the case $n = 3$. The T -fixed points are the 6 permutation matrices in S_3 . We write on the left the fixed points and their associated Bruhat cells. The entries labelled * refer to a free variable, and the fixed points are obtained by letting these entries to be zero. On the right we picture the dimension of the corresponding cell.



Hence, in this case the Betti numbers of \mathcal{F} are $b_0 = 1, b_2 = 2, b_4 = 2, b_6 = 1$, and no odd Betti numbers. This example is worked out in more detail in Example 3.5.

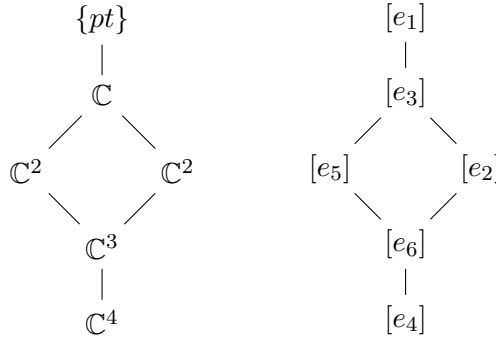
Example 2.21. Let $Gr(2, 4)$ be a Grassmannian of 2-dimensional linear subspaces in \mathbb{C}^4 . As mentioned in Example 2.13, the Plücker embedding allows us to identify $Gr(2, 4)$ with the hypersurface

$$x_1y_1 + x_2y_2 + x_3y_3 = 0$$

in \mathbb{P}^5 . Now consider the following action of $T = \mathbb{C}^*$ on $Gr(2, 4)$:

$$t \cdot (x_1, x_2, x_3, y_1, y_2, y_3) = (t^6x_1, t^3x_2, t^4x_3, y_1, t^3y_2, t^2y_3).$$

There are 6 fixed points, the points $[e_1], [e_2], \dots, [e_6] \in \mathbb{P}^5$ corresponding to the coordinate lines in \mathbb{C}^6 (cf. Example 2.18). The corresponding plus BB-cells are given displayed below. The plus cells are pictured on the left diagram, and their corresponding fixed points are drawn on the right diagram, matching them.



As a result, the Betti numbers are: $b_0 = 1, b_2 = 1, b_4 = 2, b_6 = 1$ and $b_8 = 1$, and no odd cohomology. In Example 3.6 we compute the equivariant cohomology of this variety under an action of a three-dimensional torus.

2.5. Equivariant Cohomology and GKM theory. Equivariant cohomology is a tool that incorporates the group action into the description of

the topological properties (i.e. cohomology) of a G -space. We shall give a quick overview of the subject, focusing on the case of T -skeletal varieties (Definition 2.22). For these varieties, equivariant cohomology can be described quite combinatorially from the associated graph of fixed points and invariant curves (Theorem 2.23). Hence, for our purposes, Theorem 2.23 could be taken as the definition of equivariant cohomology for T -skeletal varieties.

Let $T = (\mathbb{C}^*)^d$ be an algebraic torus. Let S denote the symmetric algebra over \mathbb{Q} of the character group Δ of T . In other words, S is isomorphic to $\mathbb{Q}[x_1, \dots, x_d]$, the polynomial ring in d variables over \mathbb{Q} . Under this isomorphism a character χ of T , with weight (a_1, \dots, a_d) , is identified with with the linear polynomial $a_1x_1 + a_2x_2 + \dots + a_dx_d \in \mathbb{Q}[x_1, \dots, x_d]$.

For a T -variety X , we denote by $i_T : X^T \rightarrow X$ the inclusion of the fixed point set. The (rational) T -equivariant cohomology of a T -variety X is denoted by $H_T^*(X)$. The T -equivariant cohomology of a point identifies to S . Thus, $H_T^*(X)$ is an algebra over S . See [Bo] and [Br1] for more details.

We recall some basic facts and nomenclature from [GKM] and [G1].

Definition 2.22. Let X be a projective T -variety. Let $\mu : T \times X \rightarrow X$ be the action map. We say that μ is a **T -skeletal action** if X^T is finite, and the number of one-dimensional orbits of T in X is finite. In this context, X is called a **T -skeletal variety**. The associated graph of fixed points and invariant curves is called the **GKM graph** of X . We shall denote this graph by $\Gamma(X)$.

Let X be a T -skeletal variety. We write $X^T = \{x_1, \dots, x_m\}$. Now, as mentioned previously, one can associate to X a ring $PP_T(X)$ of **piecewise polynomial functions**. This ring is defined using solely the combinatorial information of the graph $\Gamma(X)$. We recall the construction here. Let $H_T^*(X^T) = \bigoplus_{x \in X^T} S_x$, where S_x is a copy of the polynomial ring $S = \mathbb{Q}[x_1, \dots, x_d]$. We define $PP_T(X)$ as the subalgebra of $H_T^*(X^T) = S^m$ given by

$$PP_T(X) = \{(f_1, \dots, f_m) \in S^m \mid f_i - f_j \text{ is divisible by } \chi_{i,j}\}$$

where x_i and x_j are the two fixed points in the closure of the one-dimensional T -orbit $C_{i,j}$, and $\chi_{i,j}$ is the character of T associated with $C_{i,j}$. The character $\chi_{i,j}$ is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite).

For a T -skeletal variety X , the **GKM data** of X is the data of T -fixed points, T -stable curves, and the corresponding characters of T associated to these curves. The GKM data of X is the information needed to construct $PP_T(X)$.

Let X be a T -skeletal variety. One is interested in determining under what conditions $PP_T(X)$ actually encodes all the information of the equivariant

cohomology $H_T^*(X)$. In other words, if X is T -skeletal, can we recover geometric information about X from the combinatorial data of $\Gamma(X)$? The next result provides an answer to this question. It is due to Goresky, Kottwitz and MacPherson, and it is the main theorem in our setting.

Theorem 2.23 ([GKM, Theorem 1.2.2]). *Let X be a T -skeletal variety. If X has no cohomology in odd degrees, then the restriction map*

$$i_T^* : H_T^*(X) \rightarrow H_T^*(X^T)$$

is injective, and it induces an isomorphism between $H_T^(X)$ and $PP_T(X)$.*

A T -skeletal variety that satisfies the hypothesis of Theorem 2.23 is called a **GKM variety**.

Notation: We shall interpret the image of i_T^* as the set of all maps

$$\varphi : X^T \rightarrow S$$

such that $\varphi(x_i) - \varphi(x_j)$ is divisible by χ whenever x_i and x_j are connected by an irreducible invariant curve where T acts through the character χ .

Remark 2.24. Theorem 2.23 applies to smooth T -skeletal varieties, and more generally to T -skeletal varieties that are also T -cellular, due to the Bialynicki-Birula decomposition (Corollary 2.17).

Remark 2.25. Let X be a smooth T -skeletal variety of dimension n . If $x \in X$ is a fixed point, then there are exactly n invariant curves passing through x . This implies that the GKM graph of a smooth T -skeletal variety X is *regular* (i.e. at each vertex there are exactly $\dim(X)$ edges passing through it). See e.g. [Br1].

The following lemma asserts that if X is a GKM -variety, then $PP_T(X)$ is free module over S of rank $|X^T|$, where $|X^T|$ denotes the number of fixed points. So any basis of the S -module $PP_T(X)$ has $|X^T|$ elements.

Lemma 2.26. [GKM] *Let X be a T -variety. Suppose that X has no cohomology in odd degrees and that, for the induced T -action, X^T is finite. Then $H_T^*(X)$ is a free module over S , and its rank satisfies*

$$\text{rank}_S H_T^*(X) = \dim_{\mathbb{Q}} H^*(X) = |X^T|,$$

□

In Section 3 we shall provide concrete examples of GKM -varieties and compute, explicitly, bases for the corresponding equivariant cohomology $PP_T(X)$.

3. PROCESS TO OBTAIN THE EQUIVARIANT COHOMOLOGY

Based on the results of Section 2, given smooth T -skeletal variety X , the process to compute its equivariant cohomology $PP_T(X)$ can be summarized in the following steps.

- (1) Compute the fixed points
- (2) Calculate the invariant curves
- (3) Calculate the stabilizers (of invariant curves)
- (4) Associate the action to a graph
- (5) Assign polynomials to each edge (the weights of the corresponding invariant curves)
- (6) Obtain the associated algebraic object (by solving the imposed divisibility conditions)
- (7) Describe the equivariant cohomology

The purpose of this section is to illustrate this procedure in some concrete cases. Examples 3.1, 3.5 and 3.6 are based on [Ty]. For simplicity, we take cohomology with complex coefficients in this section.

Example 3.1. Let $T=(\mathbb{C}^*)^2$ act on \mathbb{P}^2 via:

$$(t, s).[x : y : z] = [x : ty : sz].$$

We find *all* orbits of this action, their respective stabilizers, and as a result, get the equivariant cohomology $H_T^*(X)$ [Ty]. In this example we illustrate step by step the process outlined at the beginning of this Section.

- (1) Fixed Points: $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$
- (2) Orbits when $x = 0$, or $y = 0$, or $z = 0$ (i.e. looking for orbits on the lines at infinity), or when all variables are nonzero (i.e. looking for orbits that lie on the complement of the union of the lines at infinity)
 - $x = 0$

$$O[0 : a : b] = O[0 : 1 : c] = O[0 : 1 : 1]$$

$$O[0 : a : b] = \{[x : y : z] = (t, s).[0 : 1 : 1] \mid \forall t, s \in \mathbb{C}^*\}$$

$$O[0 : a : b] = \{[x : y : z] = [0 : t : s]\}$$
 - $y = 0$

$$O[a : 0 : b] = O[1 : 0 : c] = O[1 : 0 : 1]$$

$$O[a : 0 : b] = \{[x : y : z] = (t, s).[1 : 0 : 1] \mid \forall t, s \in \mathbb{C}^*\}$$

$$O[0 : a : b] = \{[x : y : z] = [1 : 0 : s]\}$$
 - $z = 0$

$$O[a : b : 0] = O[1 : c : 0] = O[1 : 1 : 0]$$

$$O[a : b : 0] = \{[x : y : z] = (t, s).[1 : 1 : 0] \mid \forall t, s \in \mathbb{C}^*\}$$

$$O[0 : a : b] = \{[x : y : z] = [1 : t : 0]\}$$
 - $x \neq 0, y \neq 0, z \neq 0$

$$O[x : y : z] = O[1 : 1 : 1].$$
 Indeed,

$$O[1 : 1 : 1] = \{[x : y : z] \in \mathbb{P}^2 \mid [x : y : z] = (t, s) \cdot [1 : 1 : 1]\} = \{[x : y : z] = [1 : t : s]\} \forall s, t \in \mathbb{C}^*.$$

Hence, all points in the complement of the lines at infinity lie in one (open) orbit.

(3) Stabilizers. For each orbit found above, we calculate the corresponding stabilizer

- $[0 : 1 : 1]$

$$\text{Stab}[0 : 1 : 1] = \{(t, s) \in (\mathbb{C}^*)^2 \mid (t, s) \cdot [0 : 1 : 1] = [0 : 1 : 1]\}$$

$$\text{Stab}[0 : 1 : 1] = \{(t, t) \mid t \in (\mathbb{C}^*)\} \rightarrow t = s \leftrightarrow ts^{-1} = 1$$

- $[1 : 1 : 0]$

$$\text{Stab}[1 : 1 : 0] = (t, s) \cdot [1 : 1 : 0] = [1 : 1 : 0] = [1 : t : 0] \rightarrow t = 1.$$

$$\text{Stab}[1 : 1 : 0] = \{(t, s) \in (\mathbb{C}^*)^2 \mid t = 1\} = \{(1, s), s \in \mathbb{C}^*\}.$$

- $[1 : 0 : 1]$

$$\text{Stab}[1 : 0 : 1] = \{(t, s) \in (\mathbb{C}^*)^2 \mid (t, s) \cdot [1 : 0 : 1] = [1 : 0 : 1]\}$$

$$\text{Stab}[1 : 0 : 1] = \{(t, s) \in (\mathbb{C}^*)^2 \mid s = 1\} = \{(t, 1), t \in \mathbb{C}^*\}.$$

- $x \neq 0, y \neq 0, z \neq 0$

$$\text{Stab}[1 : 1 : 1] = \{(1, 1)\}. \text{ Thus is orbit is isomorphic to } T.$$

To sum up, the action of $T=(\mathbb{C}^*)^2$ on $\mathbb{C}P^2$ has seven orbits: an open orbit isomorphic to T , three invariant curves, and three are fixed points. For our description, we only need the invariant curves and fixed points. Furthermore, each curve comes with an associated stabilizer (described by a weight of T). Figure 1 encodes our findings.

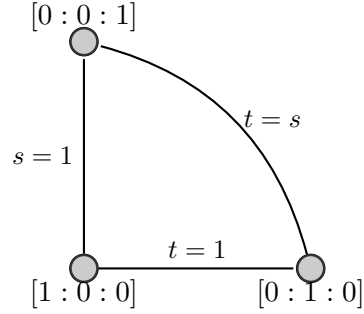


FIGURE 1. Variety X

(4) Associate the action to a graph

We get the weights corresponding to the stabilizers found in the previous step.

$$(t, s \in \mathbb{C}^*)$$

- $t = s \rightarrow ts^{-1} = 1 \rightarrow (1, -1)$

- $t = 1 \rightarrow t = 1 \rightarrow (1, 0)$

- $s = 1 \rightarrow s = 1 \rightarrow (0, 1)$

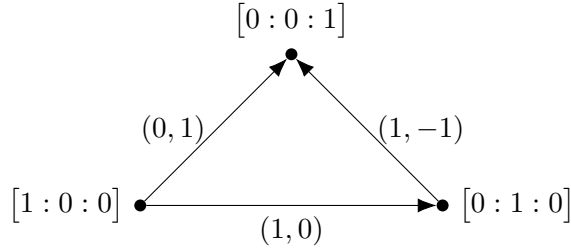


FIGURE 2. Graph associated to the action

- (5) Define polynomials for each edge
 For each weight found, associate its polynomial in $S = \mathbb{C}[x, y]$.

- $(1, -1) \rightarrow x - y$
- $(1, 0) \rightarrow x$
- $(0, 1) \rightarrow y$

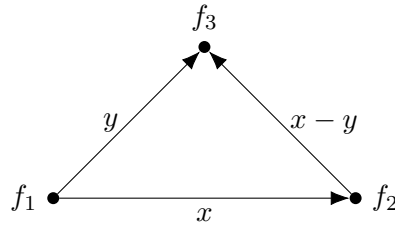


FIGURE 3. Polynomials for each edge

- (6) Obtain the algebraic object associated
 We associate to each edge a polynomial in two variables, x and y , subject to the following condition:

- $(f_2 - f_1)$ divisible by x
- $(f_3 - f_1)$ divisible by y
- $(f_3 - f_2)$ divisible by $x - y$

Then, we obtain the following polynomials:

- $f_3 = f_1 + p(x, y) \cdot y$
- $f_2 = f_1 + q(x, y) \cdot x$
- $f_3 = f_2 + s(x, y) \cdot (x - y)$

For instance, the following triples fulfill the conditions mentioned above: $(0, 0, 0)$, $(x, 0, x - y)$, $(1, 1, 1)$, $(0, x, y)$, $(0, 0, y \cdot (x - y))$.

- (7) Describe the equivariant cohomology

$$H_T^*(X) = \{(f_1, f_2, f_3) \in \mathbb{C}[x, y]\}$$

$$H_T^*(X) = \{(f_1, f_1 + p(x, y) \cdot y, f_1 + q(x, y) \cdot x)\}$$

Moreover, to find a basis B for the equivariant cohomology we need a certain number of generators

$$B = \{(1, 1, 1), (0, y, x), (0, 0, x \cdot (x - y))\}$$

To summarize, this action of $T = (\mathbb{C}^*)^2$ in \mathbb{P}^2 has a finite number of orbits, of which, 3 are curves and 3 are fixed points. Then, after we find the weights of the action, we can associate this with the decorated graph being the invariant curves its edges, and the fixed points the vertices of the graph [Ty].

Example 3.2. Let $T=(\mathbb{C}^*)^3$ act on \mathbb{P}^3 via:

$$(t, s, w).[x_0; x_1; x_2; x_3] = [x_0; tx_1; sx_2; wx_3].$$

We find all the GKM data, i.e. fixed points, one-dimensional orbits, and their corresponding stabilizers and weights. Also, for each weight we associate its corresponding polynomial in $S = \mathbb{C}[x, y, z]$ (or its algebraic representation). As a result, we exhibit the equivariant cohomology. Since the analysis for \mathbb{P}^3 is quite similar to the one given for \mathbb{P}^2 in the previous example, we skip the details. The results are shown in the following table.

TABLE 1. GKM data

Orbit	Stabilizer	Weight	Algebraic Representation
$[0 : 0 : 1 : 1]$	$sw^{-1} = 1$	$(0, 1, -1)$	$y - z$
$[0 : 1 : 0 : 1]$	$tw^{-1} = 1$	$(1, 0, -1)$	$x - z$
$[0 : 1 : 1 : 0]$	$ts^{-1} = 1$	$(1, -1, 0)$	$x - y$
$[1 : 0 : 0 : 1]$	$w = 1$	$(0, 0, 1)$	z
$[1 : 0 : 1 : 0]$	$s = 1$	$(0, 1, 0)$	y
$[1 : 1 : 0 : 0]$	$t = 1$	$(1, 0, 0)$	x

Therefore, we can represent these results in the GKM graph. Then, considering the polarizing vector $\lambda = (1, 2, 3)$ it is obtained the following sorted graph.

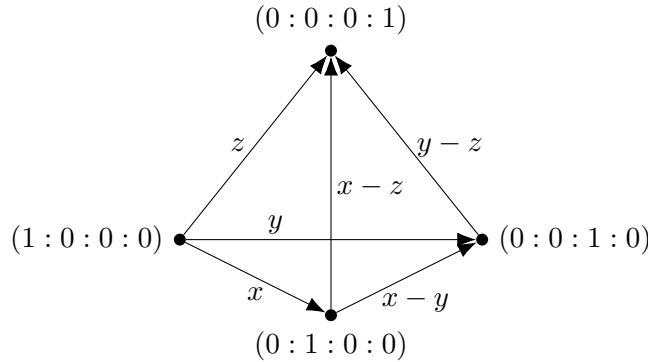


FIGURE 4. Sorted GKM Graph

From the GKM graph we can read off the divisibility conditions (the fixed points have been labelled x_1, x_2, x_3, x_4 , where the sub-index indicates the position of the only nonzero entry) and this labels are related to f_1, f_2, f_3, f_4 respectively, where it is possible to obtain their following polynomials:

- $f_3 = f_2 + p_1(x, y, z) \cdot (x - y)$
- $f_4 = f_2 + q_1(x, y, z) \cdot (x - z)$
- $f_2 = f_1 + r_1(x, y, z) \cdot (x)$
- $f_4 = f_3 + p_2(x, y, z) \cdot (y - z)$
- $f_3 = f_1 + q_2(x, y, z) \cdot (y)$
- $f_4 = f_1 + r_2(x, y, z) \cdot (z)$

Finally, if we solve the equations, we can obtain the basis B for the equivariant cohomology of the action:

- $P_4 = (0, 0, 0, z \cdot (x - z) \cdot (y - z))$
- $P_3 = (0, 0, y \cdot (x - y), z \cdot (x - z))$
- $P_2 = (0, x, y, z)$
- $P_1 = (1, 1, 1, 1)$

$$B = \{P_1, P_2, P_3, P_4\}$$

Example 3.3. Consider the following action of $T = \mathbb{C}^* \times \mathbb{C}^*$ on \mathbb{P}^3 :

$$\left(\left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right), \left(\begin{array}{cc} \beta & 0 \\ 0 & \beta^{-1} \end{array} \right) \right) \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\alpha\beta^{-1} & b\alpha\beta \\ c\alpha^{-1}\beta^{-1} & d\alpha^{-1}\beta \end{bmatrix}.$$

Here, since \mathbb{P}^3 is a quotient of $\mathbb{C}^4 \setminus \{0\}$ by the linear action of \mathbb{C}^* , it makes sense to represent a point in \mathbb{P}^3 as the class of a non-zero 2×2 matrix.

Solving the corresponding equations, we note that for a matrix to be a fixed point, it must have only one non-zero entry. So there are four fixed points:

$$x_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the one dimensional orbits we can assume that the matrix given above has at least one non-zero entry. Without loss of generality we can consider that such entry is $a = 1$. Then the other entries b, c, d cannot be all zero, for otherwise we will end up with a fixed point. So, for instance, say d is non-zero. Under these assumptions the orbit of X looks like:

$$\begin{bmatrix} \alpha\beta^{-1} & b\alpha\beta \\ c\alpha^{-1}\beta^{-1} & d\alpha^{-1}\beta \end{bmatrix}.$$

Clearly, the only way we could get a one dimensional orbit is to have b and c equal to zero. In such case, our matrix looks as follows:

$$\begin{bmatrix} \alpha\beta^{-1} & 0 \\ 0 & d\alpha^{-1}\beta \end{bmatrix}.$$

Taking into account our choices, we get six one-dimensional orbits,

$$E_1, E_2, E_3, E_4, E_5, E_6,$$

which are the orbits of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

respectively.

Next we describe the stabilizers Γ_i of the one dimensional orbits E_i , as the kernel of some characters $\chi_i : T \rightarrow \mathbb{C}^*$, $i = 1, \dots, 6$.

The orbit of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ consists of all the matrix classes of the form

$$\begin{bmatrix} \alpha\beta^{-1} & 0 \\ 0 & \alpha^{-1}\beta \end{bmatrix},$$

and thus the stabilizer Γ_1 of E_1 consists of all the pairs

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \right)$$

subject to the condition

$$\alpha^2\beta^{-2} = 1;$$

in other words, Γ_1 is the kernel of the character $\chi_1 : T \rightarrow \mathbb{C}^*$ given by

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \right) \mapsto \alpha^2\beta^{-2}.$$

Likewise, we find $\Gamma_2 = \ker(\chi_2) = \ker(\alpha^2)$, $\Gamma_3 = \ker(\chi_3) = \ker(\beta^2)$ and $\Gamma_4 = \ker(\chi_4) = \ker(\alpha^2\beta^2)$. Furthermore, $\Gamma_5 = \ker(\chi_2)$ as well as $\Gamma_6 = \ker(\chi_3)$. We thus assume that $\chi_5 = \chi_2$ and $\chi_6 = \chi_3$ in our enumeration above.

The next GKM graph describes the fixed points and the algebraic representation of the stabilizers for each one dimensional orbit. Considering the polarizing vector $\lambda = (-2, 1)$, we get the following orientation.

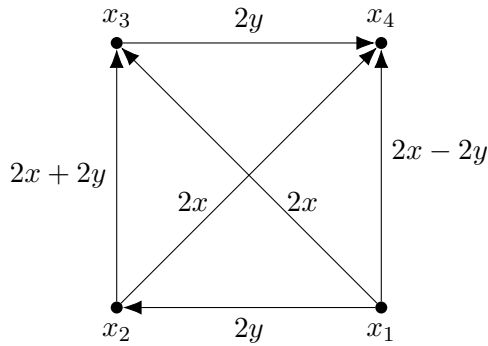


FIGURE 5. Associated Graph

Solving the congruence relations we obtain the following:

The T -equivariant cohomology of \mathbb{P}^3 for the algebraic action just described is generated as a module over $S = \mathbb{C}[x, y]$ by the following generators

- $P_1 = (1, 1, 1, 1)$

- $P_2 = (0, y, -x, -x + y)$
- $P_3 = (0, 0, x(x + y), x(x - y))$
- $P_4 = (0, 0, 0, xy(x - y))$

Example 3.4. Let \mathbb{F}_n be the Hirzebruch surface (Examples 2.11, 2.19). Let T be a two dimensional torus. We consider the natural T -action on \mathbb{F}_n given by

$$(t, s) \cdot ([x_0, x_1, x_2], [y_1, y_2]) = ([tx_0, s^n x_1, x_2], [y_1, sy_2]).$$

Note that this action fixes the following points of \mathbb{F}_n :

$$x = ([0 : 0 : 1], [1 : 0]),$$

$$y = ([0 : 1 : 0], [0 : 1]),$$

$$z = ([1 : 0 : 0], [1 : 0]),$$

and

$$w = ([1 : 0 : 0], [0 : 1]).$$

One can check that there are only finitely many curves on \mathbb{F}_n fixed by T , these are C_-, L_0, C_+, L_∞ (with the notation introduced in Example 2.11). Moreover, we have $z, w \in C_-, x, z \in L_0, x, y \in C_+, y, w \in L_\infty$

Now we look at the weights of T on these curves (recall that these weights are uniquely defined up to sign):

- the weight of T associated to C_+ is s .
- the weight of T corresponding to L_∞ is ts^{-n} .
- the weight of T associated to C_- is s .
- the weight of T associated to L_0 is t .

The corresponding GKM-graph is

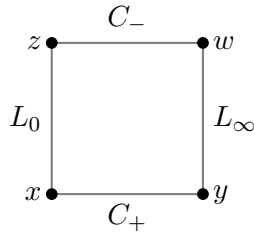


FIGURE 6. Associated Graph

Using a generic one-parameter subgroup such as $\lambda = (n, 2)$, we obtain the filtration given in Example 2.19, which induces the following order of the fixed points $x \leq z \leq w \leq y$. Thus the data can be sorted out in the following GKM Graph, where we have identified the characters t, s, ts^{-n} of T with the polynomials $x, y, x - ny$ in $\mathbb{C}[x, y]$.

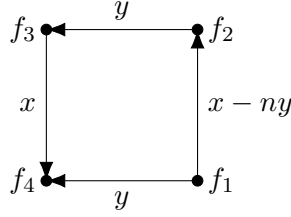


FIGURE 7. Associated Graph

We thus get the following basis of the equivariant cohomology of $\mathbb{F}(n)$:

$$\begin{aligned} f_1 &= (1, 1, 1, 1), \\ f_2 &= (0, x - ny, y + x, y), \\ f_3 &= (0, 0, y, y), \\ f_4 &= (0, 0, 0, xy) \end{aligned}$$

Example 3.5. Let \mathcal{F} be the full flag variety G/B , where $G = GL_n$ and B is the Borel subgroup of upper triangular invertible matrices (Example 2.12). In this case, $\dim \mathcal{F} = 3$.

Moreover, $x \in \mathcal{F} \Leftrightarrow \{x : V_1 \subseteq V_2 \subseteq V_3 = \mathbb{C}^3\}$, where V_1 is a line and V_2 is a plane.

Some explicit elements of \mathcal{F} are:

$$\begin{aligned} A \in \mathcal{F} &\Leftrightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{C}^3 \\ M \in \mathcal{F} &\Leftrightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{C}^3 \end{aligned}$$

For M , we have the line $V_1 = \left\{ \begin{matrix} x = y \\ z = 0 \end{matrix} \right\}$, and the plane $V_2 : z = 0$.

If we want a more compact description of the elements of \mathcal{F} , the chain can be represented using:

$$w_1 \in V_1, \{w_1, w_2\}, \{w_1, w_2, w_3\}$$

In this representation, w_1 is the generator of the line V_1 , w_2 cannot be a multiple of w_1 and w_3 is not in the plane generated by w_1 and w_2 . Hence an element of \mathcal{F} is represented by 3 linearly independent vectors and they can be thought of as the column vectors of a 3×3 invertible matrix. Namely,

$$(\vec{w}_1, \vec{w}_2, \vec{w}_3) \Leftrightarrow g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

where g is a 3×3 matrix that satisfies $\det(g) \neq 0$, that is, $g \in GL_3$.

Furthermore, we can say that:

$$\mathcal{F} \underset{\text{isomorphic}}{\Leftrightarrow} O(e) = GL_3 / \text{Stabilizer}$$

An element $[g] \in GL_3/\text{Stab}$ represents the class of elements of the form: $g \cdot h$, where $h \in \text{Stab}(e)$, and e is a canonical flag:

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The isomorphism is given by the map:

$$\begin{aligned} G &= GL_3 \\ G &\xrightarrow{\varphi} \mathcal{F} \\ g &\longmapsto g \cdot e \end{aligned}$$

Here, φ is surjective because any $x \in \mathcal{F}$ can be reached from e by using g_x , the invertible matrix in G associated to x . Indeed, we have

$$g_x \cdot e = x$$

However, φ is not injective because if A is an upper triangular matrix we have: $A \cdot e = e$. In other words, φ collapses the upper triangular matrices. Hence, the map φ factors through the quotient space G/B giving the desired isomorphism between G/B and \mathcal{F} (Cf. Remark 2.4). (Certainly, one should also check that φ is an isomorphism of algebraic varieties. For a proof of this last step, see [Hu]).

Let B denote the subgroup of G consisting of upper triangular matrices. Let B^- denote the subgroup of G formed by the lower triangular matrices. We can make B act on the right of \mathcal{F} and B^- on the left of \mathcal{F} . In fact, since \mathcal{F} is isomorphic to G/B , the left action of B^- allows to decompose or partition \mathcal{F} into B^- -orbits. This leads to the Bruhat decomposition:

\mathcal{F} = union of orbits of the action of B on G by right and left multiplication.

$$\mathcal{F} \stackrel{\text{Bruhat}}{=} \bigsqcup_{s \in S_n} B \cdot s \cdot B^-$$

where s is a permutation matrix.

Now consider the action of $T = (\mathbb{C}^*)^2$ on \mathcal{F} given as follows:

If $t = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t_2 \end{pmatrix} \in T$ and $g_{ij} \in \mathcal{F}$, then $t \star g$ is the element of \mathcal{F} given by

$$\begin{aligned} t \star g &:= t \cdot g \cdot t^{-1} = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t_2 \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \cdot \begin{pmatrix} t_1^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t_2^{-1} \end{pmatrix} \\ t \star g &:= \begin{pmatrix} g_{11} & t_1 \cdot g_{12} & t_1 \cdot t_2^{-1} \cdot g_{13} \\ t_1^{-1} \cdot g_{21} & g_{22} & t_2^{-1} \cdot g_{23} \\ t_2 \cdot t_1^{-1} \cdot g_{31} & t_2 \cdot g_{32} & g_{33} \end{pmatrix} \end{aligned}$$

It follows from the Bruhat decomposition that the T -fixed points of \mathcal{F} are the permutation matrices. On the other hand, the invariant curves are parametrized by the matrices which are a permutation matrix plus a

single nonzero entry a . Moreover, the cells are obtained by placing zeros to the right and below the nonzero entries and leaving the other entries free. Hence, to get an invariant curve we can place a parameter a on each free entry (cf. Example 2.20).

For this action, note that the associated weights of T are $x, y, x - y$. Next we choose the generic one parameter subgroup $\lambda = (t^{-1}, t^{-2})$, equivalently we choose the polarizing vector $(-1, -2)$.

Observe that $T = (\mathbb{C}^*)^2$ acts on $\tilde{s} = \begin{pmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{F}$ via

$$t \star \tilde{s} = \begin{pmatrix} a & 0 & t^{-1} \\ 0 & 1 & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}$$

The orbit mentioned above represents a stable curve, and we can obtain the fixed points of the graph if we let t approach 0 or ∞ . However, when we let $t \rightarrow \infty$ we can not obtain a line or a plane, so we scale the original vector (first column) to a line (i.e. we need to choose a different representative of the orbit).

Right away we note that there seems to be a contradiction, since a row full of zeros does not represent anything in this context and also, it has to be a permutation matrix. Therefore we replace the third column with the cross product of the first and second column.

$$\left\{ \begin{pmatrix} a \\ 0 \\ t^{-1} \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} a \\ 0 \\ t^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} a \\ 0 \\ t^{-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} t^{-1} \\ 0 \\ 0 \end{pmatrix} \right\} = \mathbb{C}^3$$

Let $\vec{\eta}$, Col_1 and Col_2 be:

$$Col_1 = \begin{pmatrix} a \\ 0 \\ t^{-1} \end{pmatrix}, Col_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{\eta} = Col_1 \times Col_2 = (-t^{-1}, 0, a)$$

Thus, the orbit that tends to infinity can be represented as:

$$t \rightarrow \infty \Rightarrow t_\infty = \begin{pmatrix} a & 0 & -t^{-1} \\ 0 & 1 & 0 \\ t^{-1} & 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

Also, to find the the orbit that tends to zero, we follow the same procedure as done before:

$$t \rightarrow 0 \Rightarrow t_0 = \begin{pmatrix} a \cdot t & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & a \cdot t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Considering $a = 1$, this movement can be represented as showed in the image below.

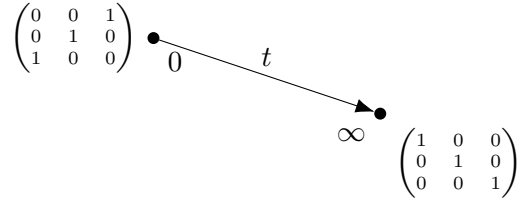


FIGURE 8. Orientation of the movement

From this procedure it can be stated that the orientation of the graph goes from top to bottom regarding to the values of the vector λ and its action in the permutation matrices. The complete filtration using this polarizing vector is shown in Example 2.20. Considering other matrices such as:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ a & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and following the same method as before, we obtain the incoming graph.

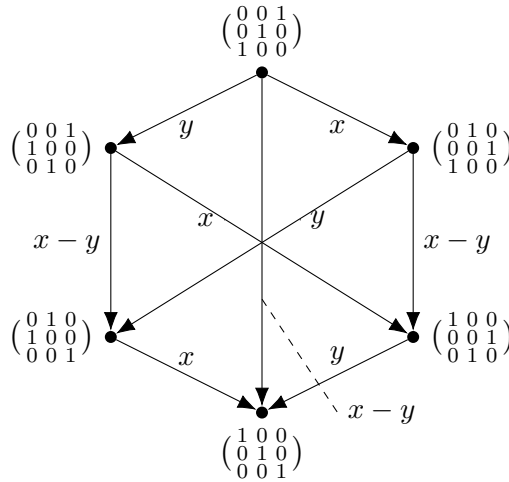


FIGURE 9. Full Flag Variety Graph or Bruhat Graph

In this example, we use the algorithm proposed in the section 4 in order to obtain the respective basis of the Full Flag Variety. The sorted graph is labeled considering the incoming edges. A detailed view of the graph is shown below, where f_1 is the lowest node since it has no incoming edges.

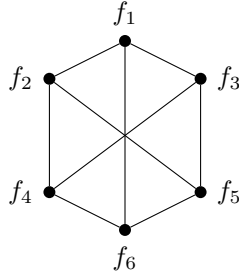


FIGURE 10. Sorted graph

Hence, to find the P_i generator associated to f_i is necessary to follow two steps. First, all lower entries are equal to zero, then the entry i is the multiplication of all incoming edges. Secondly, we have to find the values of the next entries. Each one will be the multiplication of each incoming edge with an associated zero entry. In case of another node has the same number of incoming edges, it will be equal to zero. After choosing the value of the entry, we have to validate if it satisfies the other relations. If not, we change it with its negative.

In the figure below, we show the development of this algorithm only for f_4 and f_2 . The images as displayed from left to right.

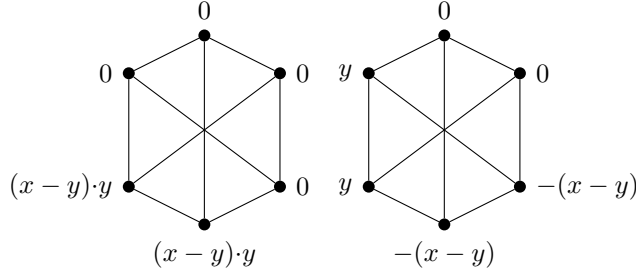


FIGURE 11. Development to find the basis of the graph

Finally, it is found the basis of $PP(\Gamma)$ as a module over $\mathbb{C}[x, y]$ which is given by the columns of the following matrix:

$$B = \begin{pmatrix} xy(xP_6 y) & x(xP_5 y) & y(xP_4 y) & (xP_3 y) & -P_2 + yP_1 \\ 0 & x(x-y) & 0 & x & -x+y & 1 \\ 0 & 0 & y(x-y) & (x-y) & y & 1 \\ 0 & 0 & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 3.6. Consider the hypersurface $X : x_1y_1 + x_2y_2 + x_3y_3 = 0$ in \mathbb{P}^5 , from Example 2.21. Let the torus $T=(\mathbb{C}^*)^3$, with coordinates (t_1, t_2, t_3) , act on X via

$$(t_1 \cdot t_2 \cdot t_3 \cdot x_1, t_1 \cdot t_2 \cdot x_2, t_1 \cdot t_3 \cdot x_3, y_1, t_3 \cdot y_2, t_2 \cdot y_3).$$

Find all the action orbits and their respective stabilizers and as a result, get the equivariant homology $H_T^*(X)$ [Ty].

According to the action, a fixed point only can have one entry different to 0. Furthermore, a one-dimensional orbit has at most two nonzero entries, but when x_i and y_j are chosen, i can not be equal to j , otherwise the orbit will be two-dimensional. In the table 2, we can see all the orbits, stabilizers, weight and the respective algebraic representations.

TABLE 2. GKM data

Orbit	Stabilizer	Weight	Algebraic Representation
[1 : 1 : 0 : 0 : 0 : 0]	$t_3 = 1$	(0, 0, 1)	z
[1 : 0 : 1 : 0 : 0 : 0]	$t_2 = 1$	(0, 1, 0)	y
[1 : 0 : 0 : 0 : 1 : 0]	$t_1 t_2 = 1$	(1, 1, 0)	$x + y$
[1 : 0 : 0 : 0 : 0 : 1]	$t_1 t_3 = 1$	(1, 0, 1)	$x + z$
[0 : 1 : 1 : 0 : 0 : 0]	$t_2 t_3^{-1} = 1$	(0, 1, -1)	$y - z$
[0 : 1 : 0 : 1 : 0 : 0]	$t_1 t_2 = 1$	(1, 1, 0)	$x + y$
[0 : 1 : 0 : 0 : 0 : 1]	$t_1 = 1$	(1, 0, 0)	x
[0 : 0 : 1 : 1 : 0 : 0]	$t_1 t_3 = 1$	(1, 0, 1)	$x + z$
[0 : 0 : 1 : 0 : 1 : 0]	$t_1 = 1$	(1, 0, 0)	x
[0 : 0 : 0 : 1 : 1 : 0]	$t_3 = 1$	(0, 0, 1)	z
[0 : 0 : 0 : 1 : 0 : 1]	$t_2 = 1$	(0, 1, 0)	y
[0 : 0 : 0 : 0 : 1 : 1]	$t_2 t_3^{-1} = 1$	(0, 1, -1)	$y - z$

In order to sort the associated graph, we are considering the polarizing vector $\lambda = (1, 2, 3)$. Therefore, the resulting action is

$$(t^6 x_1, t^3 x_2, t^4 x_3, y_1, t^3 y_2, t^2 y_3).$$

For more detail about the filtration of this action, see Example 2.21. Now, considering all orbits, stabilizers and the vector λ , we can obtain the following sorted graph:

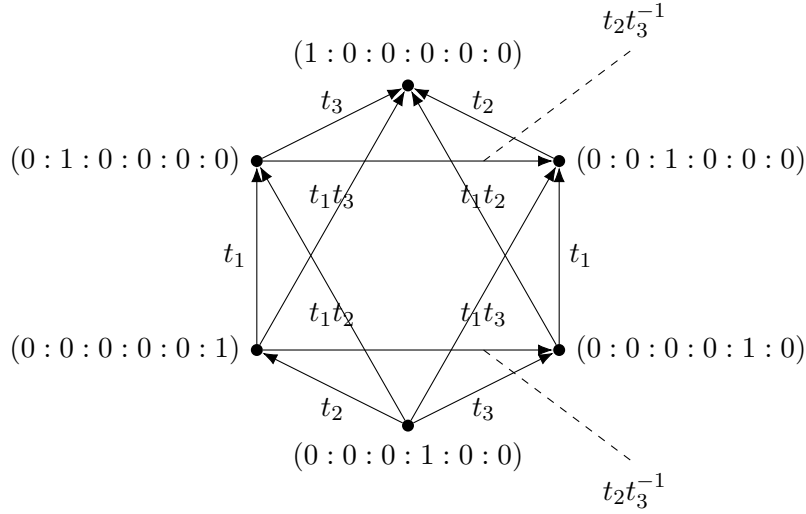


FIGURE 12. Associated Graph

The congruence relations were made such as the previous examples and it can be decodified following the values indicated in the next graph. For instance, $f_6 - f_3$ is divisible by z , $f_6 - f_4$ is divisible by y and so on.

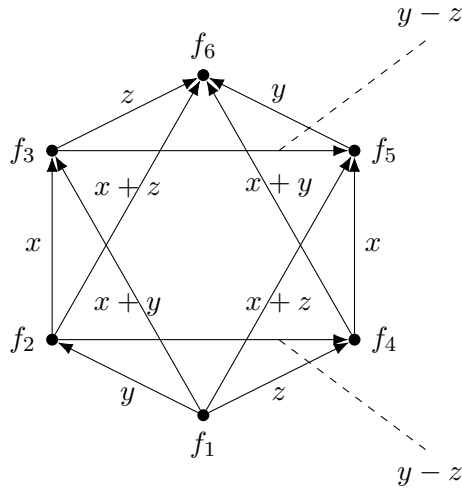


FIGURE 13. Sorted Graph

In this example, we will explain in detail how the algorithm works to generate the generator P_3 . Let E_i be the entry i of the generator, we have $P_3 = (E_1, E_2, E_3, E_4, E_5, E_6)$. The algorithm starts matching each lowest entry to 0, so for now we have: $P_3 = (0, 0, E_3, E_4, E_5, E_6)$. In order to get E_3 it is easy to see that this entry is equal to $x(x+y)$, a simple multiplication of all incoming edges. Now, due to f_4 has the same number of incoming edges than f_3 , E_4 will be zero. The process to get E_6 and E_5 is identical, so we are only going to explain the process related with E_5 . This entry only has two

options, $x(x+z)$ or $-x(x+z)$, otherwise, it does not satisfy the divisibility with E_4 and E_1 . In some cases, both options could be a valid solution, but in this opportunity the first is the unique that satisfy the divisibility relation with E_3 . In other words, first we get the options with the incoming edges with associated zero entry and then, we choose the option that satisfy the divisibility with incoming edge with associated non-zero entry. As a result of the proposed algorithm, $P_3 = (0, 0, x(x+y), 0, x(x+z), (x+y)(x+z))$.

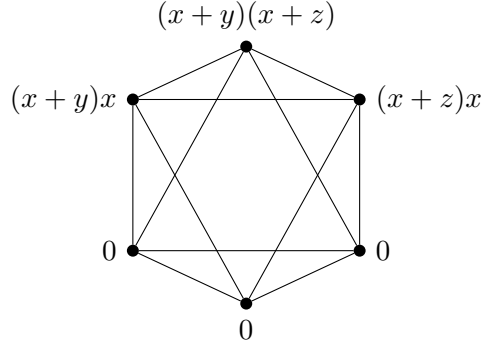


FIGURE 14. Development to find P_3

To conclude, we obtain the basis of the equivariant cohomology of the action for the next generators using the presented algorithm.

- $P_6 = \{0, 0, 0, 0, 0, (x+y)(x+z)yz\}$
- $P_5 = \{0, 0, 0, 0, (y-z)(x+z)x, (x+y)(x+z)z\}$
- $P_4 = \{0, 0, 0, (y-z)z, (x+z)(y-z), -(x+z)z\}$
- $P_3 = \{0, 0, (x+y)x, 0, (x+z)x, (x+y)(x+z)\}$
- $P_2 = \{0, y, x+y, z, x+z, x+y+z\}$
- $P_1 = \{1, 1, 1, 1, 1, 1\}$

4. ALGORITHM

The present research gives an exhaustive description of the mathematical process to find a basis of the equivariant cohomology of different varieties, which was useful to model the proposed algorithm. It is divided into three sub modules, get a polarizing vector (see Fig. 16), sort graph (see Fig. 17) and get a basis of equivariant cohomology (see Fig. 18, 19 and 20).

The modeling of this algorithm uses the procedure carried out in the examples described in the section 3. However, we have identified that it does not work correctly in smooth non-stratifiable varieties. Thus, we will continue this project in order to generalize the algorithm to this kind of varieties and to include singular projective varieties.

To use the algorithm, the user have to give us the torus dimension, the action, the one-dimensional orbits and the vertices. The torus dimension provides a number of variables, the graph data allow us to choose a polarized vector and the action determines the orientation.

Let n be the number of vertices and m the dimension of a variety X , the exact implementation of the proposed algorithm has a computational complexity of $O(m * n^2)$. However, one way to reduce significantly the execution time is using a thread for each generator, it is possible because each one is independent of each other.

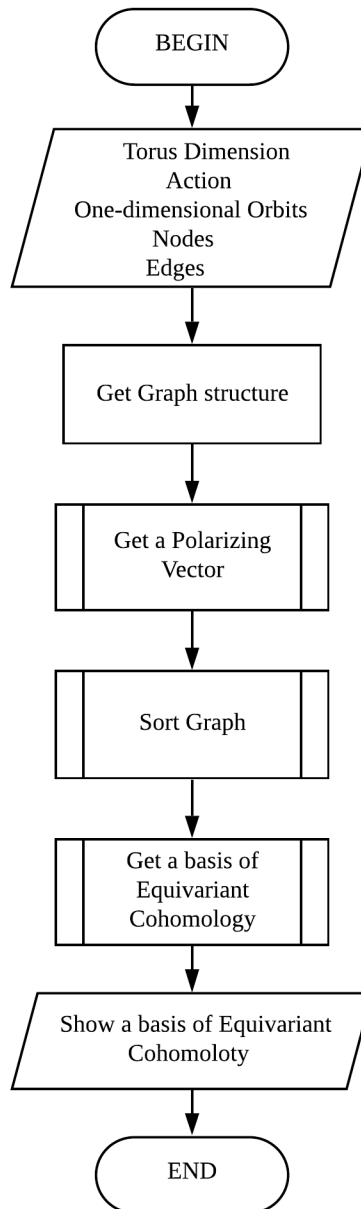


FIGURE 15. Main Program

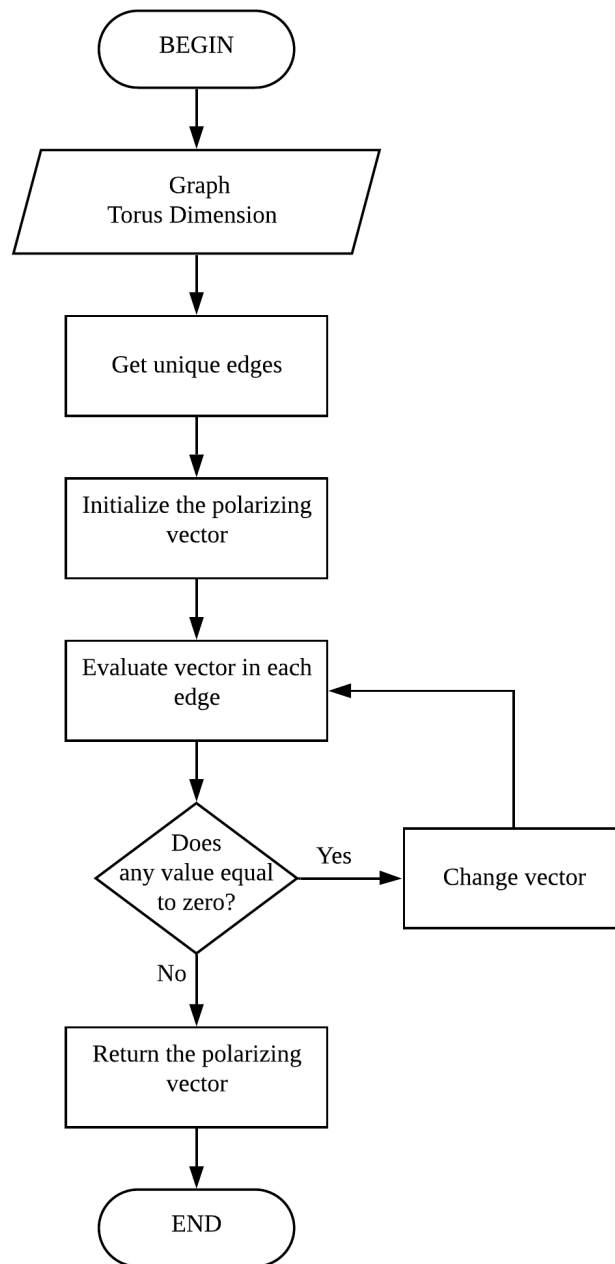


FIGURE 16. Get Polarizing Vector

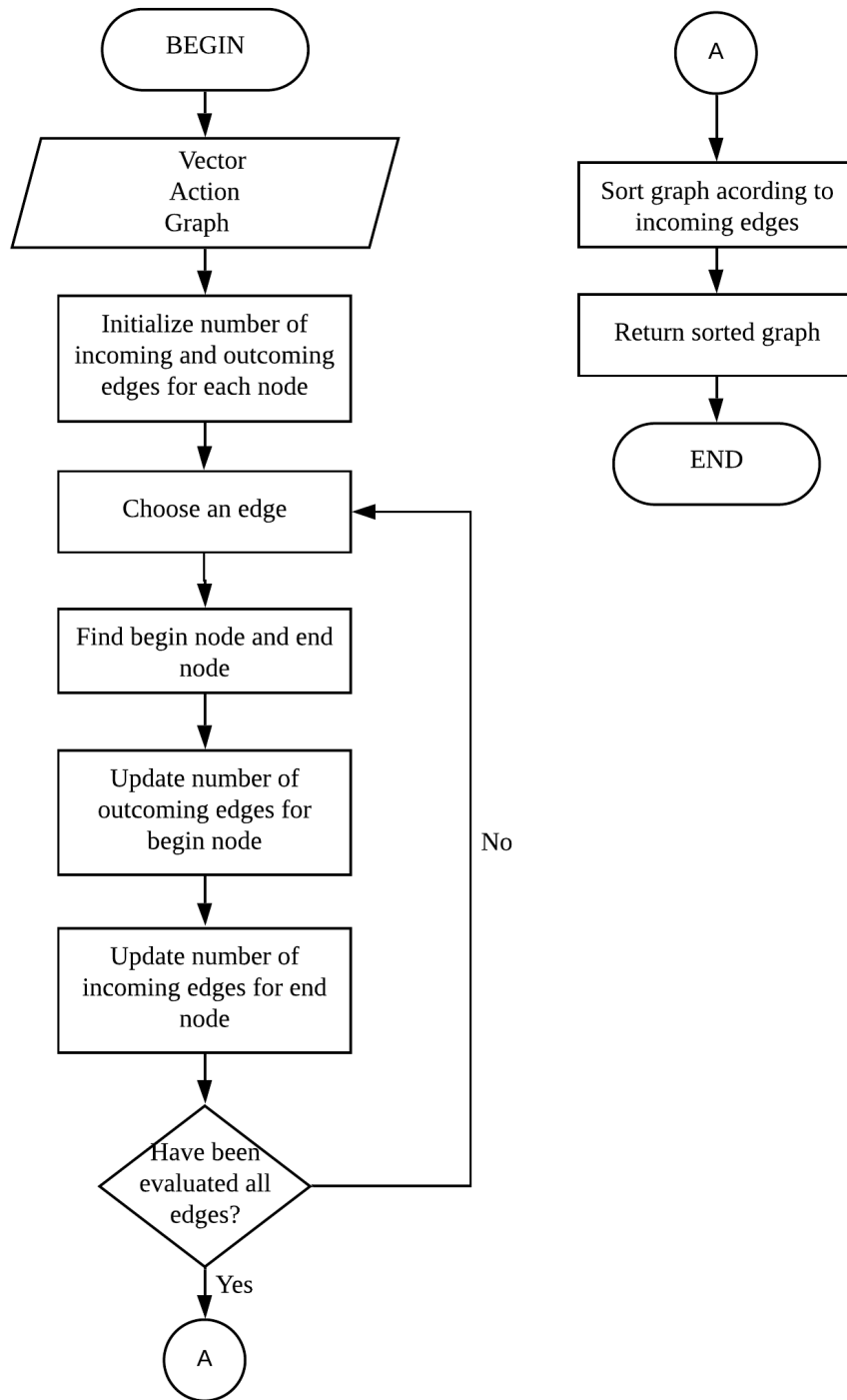


FIGURE 17. Sort Graph

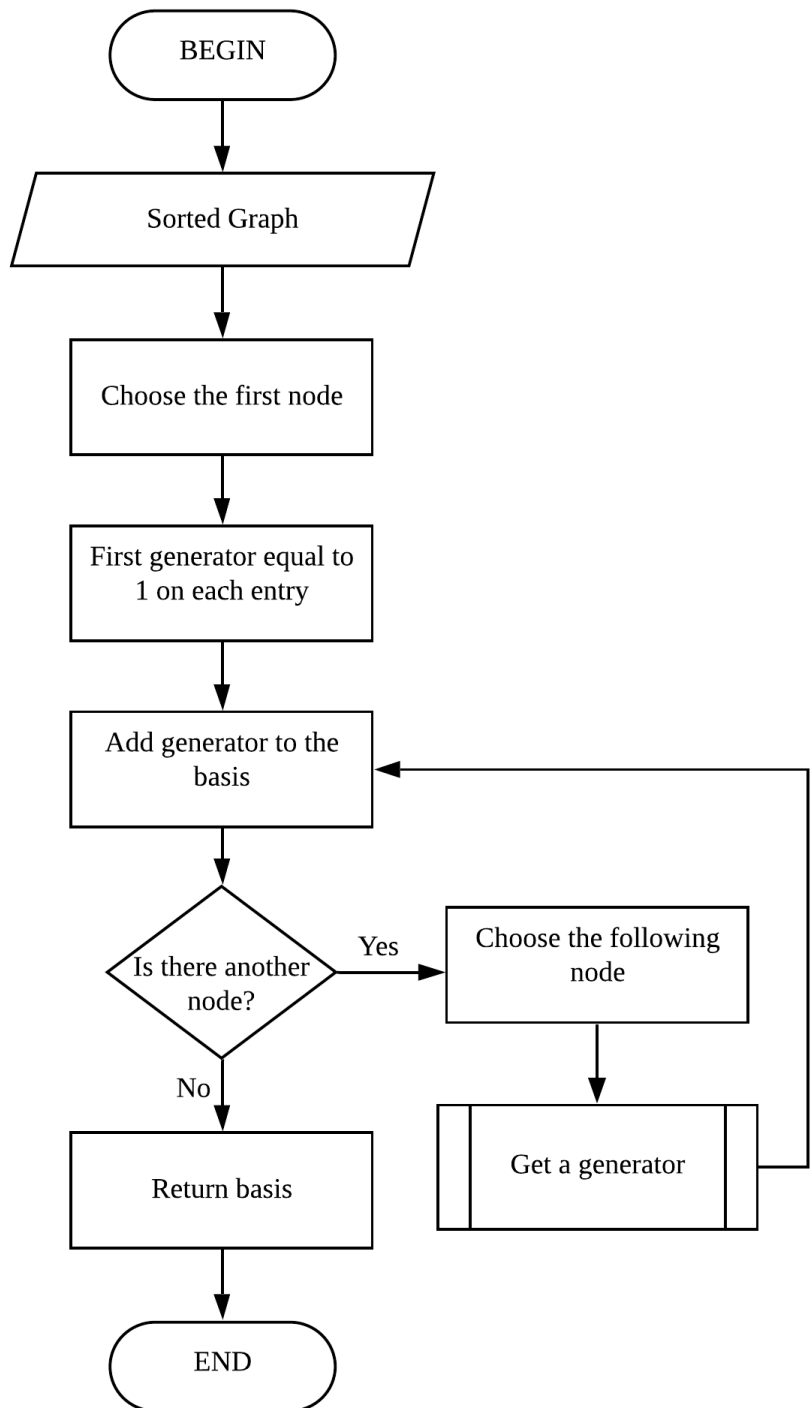


FIGURE 18. Get a basis of Equivariant Cohomology

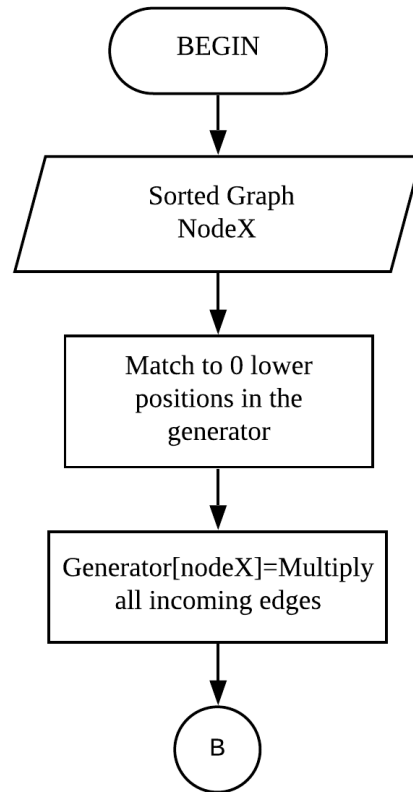


FIGURE 19. Get a generator

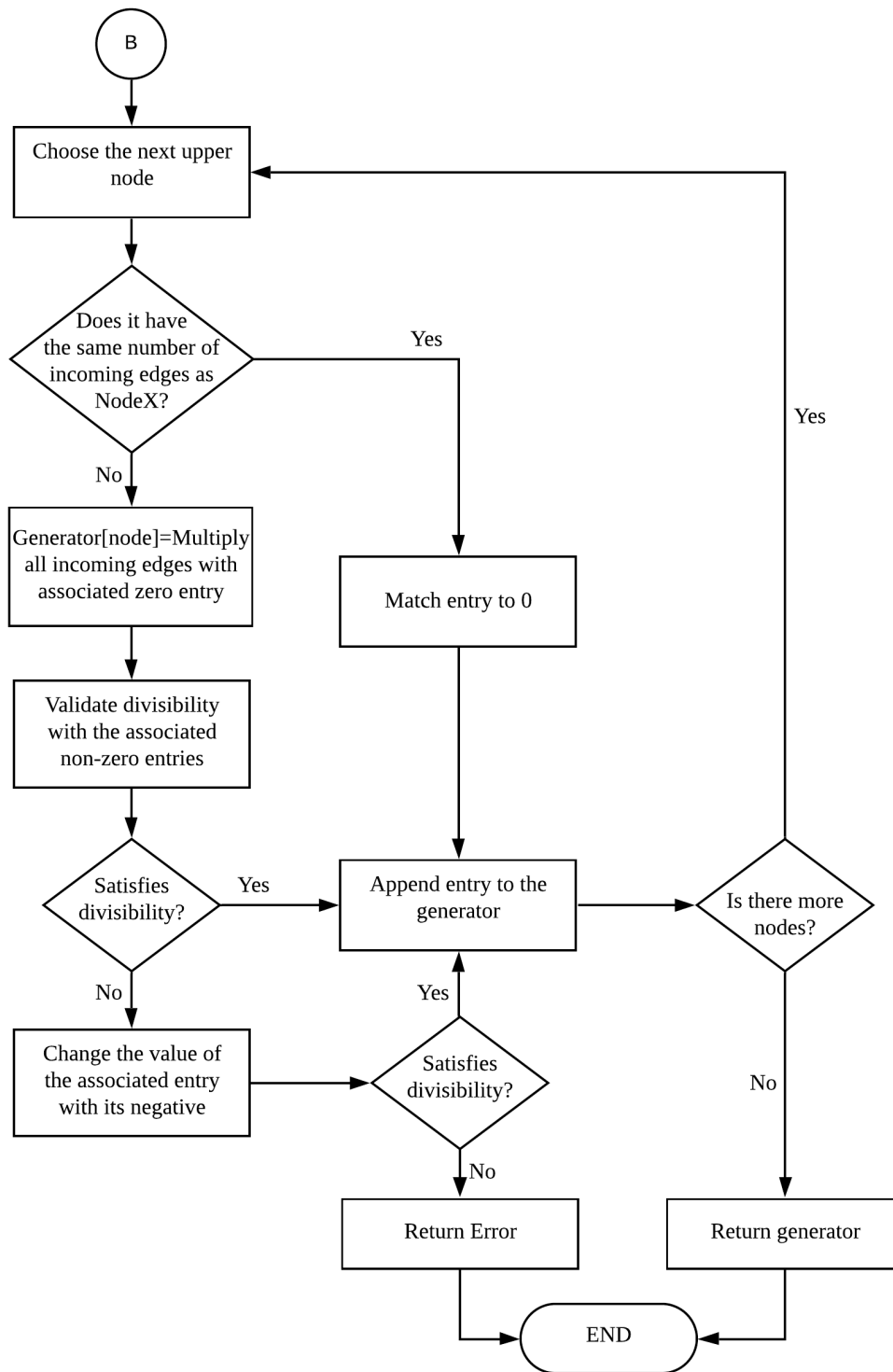


FIGURE 20. Get a generator

To conclude, we present the progress of the implementation of the algorithm. We introduce then the "getVector" function and "Graph" class structure. Furthermore, we use sympy library to evaluate and operate polynomials. The final results and the related code will be published in the near future.

```

import sympy
from sympy import *

class Graph:

    def __init__(self, nodes, weights, symbols):
        self.sort=[]
        self.weights=weights
        self.symbols=symbols
        self.nodes=dict()
        for i in nodes:
            self.nodes[i]=[]

    def add(self, node1, node2, weight, orbit):
        self.nodes[node1].append([node2, weight, orbit])
        self.nodes[node2].append([node1, weight, orbit])

    def getNodes(self):
        return self.nodes

    def print(self):
        for i in self.nodes:
            print(i, self.nodes[i])

def getVector(graph, torusDim):
    weights=graph.weights
    vector=dict()
    j=1
    for i in graph.symbols:
        vector[i]=j
        j+=1

    l_keys=list(vector.keys())
    condNum=len(weights)

    iterNum=0
    while (i!=condNum):

```

```

i=0
iterNum+=1
print (iterNum)
for j in range(len(weights)):
    if (int(weights[j].evalf(subs=vector))!=0):
        i+=1
    else :
        ind=iterNum%torusDim
        vector[l_keys[ind]]=vector[l_keys[ind]]*2
    break
return vector

```

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